The Measurable Space of Stochastic Processes

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***Abstract*—We introduce a stochastic extension of CCS endowed with a structural operational semantics expressed in terms of measure theory. The set of processes is organised as a measurable space by the σ-algebra generated by structural congruence classes. The structural operational semantics associates to each process an indexed class of measures over the space of processes. These measures encode the rates of the transitions from a process (state of a system) to a measurable set of processes (set of states). We prove that stochastic bisimulation is a congruence that extends structural congruence. The advantage of our calculus is that, in addition to an elegant operational semantics, it provides a natural way to define a class of metrics on processes that measure how similar two processes are in terms of behaviour. We show that these metrics characterize the stochastic bisimulation: two processes are bisimilar iff they are at distance zero; two processes are close when their stochastic behaviours are very similar.**

I. Introduction

Process algebras (PAs) [3] are formalisms designed for describing the evolution of concurrent communicating systems. For capturing observable behaviors, PAs are conceptualised along two orthogonal axes. From an algebraic point of view, they are endowed with construction principles in the form of algebraic operations that allow composing larger processes from more basic ones; a process is identified by its algebraic term. On the other hand, there exists a notion of nondeterministic evolution, described by a coalgebraic structure, in the form of a transition system. The algebraic and coalgebraic structures are not independent: Structural Operational Semantics (SOS) defines the behavior of a process inductively on its syntactic structure. In this way, classic PAs are supported by an easy and appealing underlying theory that guarantees their success.

In the past decades *probabilistic* and *stochastic* behaviors have also become of central interest due to the applications in performance evaluation and computational systems biology. *Stochastic process algebras* such as TIPP [16], PEPA [18], [19], EMPA [4] and stochastic π-calculus [29] have been defined as extensions of classic PAs, by considering more complex coalgebraic structures. The label of a stochastic transition contains, in addition to the name of the action, the rate of an exponentially distributed random variable that characterizes the duration of the transition. Consequently, SOS associates a non-negative rate value to each tuple $\langle$*state, action, state*$\rangle$. This additional information imposes important modification in the structure of the SOS format, such as the *multi-transition system* approach of PEPA or the *proved SOS* approach of stochastic π-calculus, mainly due to the fact that nondeterminism is replaced by the race policy.

With the intention of developing a stochastic process calculus for applications in computational systems biology, in this paper we propose a stochastic version of CCS [26] based on the *mass action law* [6] and equipped with an operational semantics that is particularly suited to a domain where an equational theory and a measure of similarity of behaviours is important. In the same time we aim to avoid the complicated labeling and counting of previous approaches and to provide an operational semantics that resembles the ones for nondeterministic process algebras, by lifting process-results to measure-results. For doing this, our SOS rules are not given in the pointwise style, but using constructions based on measure theory. We organise the set of processes as a measurable space and associate to each process a set of measures on it, indexed by actions. Thus, for an action a, a process P and a measurable set S of processes, the measure μ_a associated to a process P specifies the rate $\mu_a(S) \in \mathbb{R}^+$ of a-transitions from P to (elements of) S. In this way, difficult instance-counting problems that otherwise require complicated versions of SOS can be solved by exploiting the properties of measures (e.g. additivity). Similar ideas have been proposed for probabilistic automata [24], [31] and Markov processes [21], [7], [28]. Following the *transition-systems-as-coalgebras* paradigm [10], [30], this approach follows naturally in the sequence started by nondeterministic and probabilistic transition systems.

The novelty of our approach derives firstly from the structure of the measurable space of stochastic processes that we consider. The space of processes is organised by *structural congruence*, an equivalence that equates terms representing processes that we do not want to differentiate from a modeling perspective. For instance, if we model the parallel evolution of two processes, say Q and R, we expect no difference between $Q|R$, $R|Q$ and $R|Q|0$ (where 0 denotes an inactive process). This relation is required in the application domain mentioned above, where it models chemical mixing: structural congruence was invented in the first place from a chemical analogy [2]. In effect, our σ-algebra of processes is generated by the structural congruence classes meaning that the sets closed under structural congruence are the measurable sets. Our stochastic transitions are defined from processes to measurable sets of processes. In this way, if a process P can perform an action a with a rate r to $Q|R$, written $P \xrightarrow{a,r} Q|R$, we can also derive $P \xrightarrow{a,r} R|Q$ and $P \xrightarrow{a,r} R|(Q|0)$. Otherwise, the alternative approach of considering any set of processes measurable permits to calculate the rate of the a-transitions from P to the set $\{Q|R, R|Q, R|(Q|0)\}$ and obtain the undesired result $P \xrightarrow{a,3r} \{Q|R, R|Q, R|(Q|0)\}$; to avoid such problems, in the literature have been proposed complicated variants of SOS that make the underlying theory heavy and problematic.

Our choice of developing a stochastic process algebra under the restrictions of the equational theory induced by structural congruence is sustained by an elegant SOS that supports a smooth development of the basic theory and the definition of metrics for stochastic behaviour. The structures we obtain, simply called *Markov processes* (MPs), are particular cases of continuous Markov processes defined in [14]; they extend the notions of *labelled Markov process* [5], [13], [12] and *Harsanyi type space* [17], [20], [27] on to the stochastic level. However, MPs are not continuous-time Markov chains because each transition is from a state to an infinite class of states (closed to the equational theory) and consequently cannot be described in a pointwise style.

We also introduce a notion of *stochastic bisimulation* for MPs, along the lines of [25], [13], [12], [14]. The stochastic bisimulation generalizes *rate aware bisimulation* introduced in [9], being defined for arbitrary measurable spaces and closed with respect to an equational theory (defined, in our case, by structural congruence). We prove that, for our process algebra, stochastic bisimulation is a congruence that extends structural congruence.

An other advantage of our approach consists in the fact that it can be naturally extended to define a class of metrics on stochastic processes which measure the similarity of process behaviours. This result has considerable practical application. The standard notion of bisimulation for probabilistic or stochastic systems cannot distinguish between two processes that are substantially different and two processes that differ by only a small amount in a real valued parameter. It is often more useful to say how similar two processes are than to say whether they are exactly the same. This is precisely what our metrics do: stochastic bisimilar processes are at distance zero, processes that differ by small values of rates are closer than the processes with bigger differences in the rate values.

The paper is organised as follows. A preliminary section establishes the basic concepts and notations. Section III defines the general concept of Markov process (MP) and the stochastic bisimulation of MPs. Section IV introduces the syntax of the minimal process algebra and the axiomatization of structural congruence; in this section we prove that the space of processes can be organised as a Markov kernel meaning that each process is an MP. The observation that the processes of our calculus are MPs guides us, in Section V, to the definition of a structural operational semantics which induces a notion of behavioural equivalence that coincides with the bisimulation of MPs. In Section VI we show that the bisimulation behaves well with respect to the algebraic structure of processes: stochastic bisimulation is a congruence. This relation is extended in Section VII with a class of metrics on the space of processes that measure how similar two processes are; two processes are at distance zero iff they are bisimilar. We also have a section dedicated to related work and a concluding section. The proofs of some of the results are collected in Appendix.

II. Preliminaries

In this section we recall a few notions of measure theory to establish the terminology and the notations used in the paper.

For arbitrary sets A and B, 2^A denotes the powerset of A, $A \uplus B$ their disjoint union and both $[A \to B]$ and B^A will be used to denote the class of functions from A to B. If $f \in B^A$ we denote by $f^{-1} : 2^B \to 2^A$ the inverse mapping of f. For an arbitrary function $f : A \to B$, the *kernel of* f is the relation $ker(f) = \{(x, y) \in A \times A \mid f(y) = f(x)\}$.

As usual $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the naturals, rationals and reals, respectively.

Given a set M, a *σ-algebra* Σ over M is a set of subsets of M containing M and closed under complement and countable union. The tuple (M, Σ) is called a *measurable space*, the elements of Σ *measurable sets* and M the *support-set*.

A set $\Omega \subseteq 2^M$ is a *generator for the σ-algebra* Σ on M if Σ is the closure of Ω under complement and countable union; we write $\overline{\Omega} = \Sigma$ and say that Σ is generated by Ω. A generator Ω for Σ is *a base of* Σ if it has disjoint elements.

A *measure* on a measurable space $\mathcal{M} = (M, \Sigma)$ is a function $\mu : \Sigma \to \mathbb{R}^+$ such that $\mu(\emptyset) = 0$ and for any $\{N_i | i \in I \subseteq \mathbb{N}\} \subseteq \Sigma$ with pairwise disjoint elements, $\mu(\bigcup_{i \in I} N_i) = \sum_{i \in I} \mu(N_i)$. The *null measure* on (M, Σ) is the measure ω such that $\omega(M) = 0$.

If Ω is a base for (M, Σ), $N \in \Omega$ and $r \in \mathbb{R}^+$, then the function $f : \Omega \to \mathbb{R}^+$

$$f(N') = \begin{cases} r & \text{if } N' = N \\ 0 & \text{if } N' \neq N \end{cases}$$

can be extended, by $f(\cup_{i \in I} N_i) = \sum_{i \in I} f(N_i)$, to a measure on (M, Σ) denoted by $D(r, N)$ and called the *r-Dirac measure on N*.

Let $\Delta(M, \Sigma)$ be the class of measures on (M, Σ). We organize it as a measurable space by considering the σ-algebra generated, for arbitrary $S \in \Sigma$ and $r > 0$, by the sets $\{\mu \in \Delta(M, \Sigma) : \mu(S) \geq r\}$.

Given two measurable spaces (M, Σ) and (N, Θ), a mapping $f : M \to N$ is *measurable* if for any $T \in \Theta$, $f^{-1}(T) \in \Sigma$. We use $[\![M \to N]\!]$ to denote the class of measurable mappings from (M, Σ) to (N, Θ).

Given a set X, a *pseudometric on* X is a function $d : X \times X \to \mathbb{R}^+$ such that
1. $\forall x \in X$, $d(x, x) = 0$;
2. $\forall x, y \in X$, $d(x, y) = d(y, x)$;
3. $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(y, z)$.
It is *a metric on* X if, in addition, satisfies
4. $\forall x, y \in X$, if $d(x, y) = 0$, then $x = y$.
If d is a metric, then (X, d) is a *metric space*.

Observe that given a pseudometric on X, one can define an equivalence on X by pairing the elements at distance zero.

The notion of an *analytic space* is central to the definition of Markov kernel and Markov process presented in the next section and is, in fact, a necessity for technical reasons. We have chosen this level of generality because our process calculus requires a more general concept than continuous-time Markov chain. For this, we propose a general definition that encapsulates most of the known concepts of Markovian systems and will allow, in future, the extension of this work toward a general algebra of Markovian processes. However, these technicalities do not influence the general presentation and for this reason we only sketch the main definitions. For a detailed discussion the reader is referred to [28] (Section 7.5) or to [15] (Section 4.4).

A metric space (X, d) is *complete* if every Cauchy sequence converges in X.

A *Polish space* is the topological space underlying a complete metric space with a countable dense subset. Note that any discrete space is Polish.

An *analytic space* is the image of a Polish space under a continuous function from one Polish space to another. Note that any Polish space is analytic. Because analytical spaces are topological spaces, they are also measurable spaces if we consider the Borel σ-algebra generated by the topology.

III. Continuous Markov processes

Before introducing the stochastic process algebra, we define a general notion of *Markov process* (MP) that encapsulates various notions of Markovian stochastic systems such as (discrete space) *Markov chain* with discrete or continuous time [21], *labelled Markov process* [28] as well as the most general case of *continuous-space and continuous-time Markov process* introduced in [14]. The notion of MP relies on the observation that a Markovian process is essentially a coalgebraic structure that encodes stochastic behaviors and can be seen, following the *transition-systems-as-coalgebras* paradigm [30], [10], as a generalisation of the notion of transition system: a transition system associates to each state of a system an action-indexed set of functions over the state space; functions with *boolean values* define *labelled transition systems* while *probabilistic distributions* define labelled Markov processes [5], [13], [12], [28] and *Harsanyi type spaces* [17], [27]. This paradigm is particularly appropriate when one is interested in systems with complex state space where transitions cannot be represented from one state to another, but from a state to a measurable set of states or to a (topological) neighbourhood.

An MP involves a set A of labels. The labels $\alpha \in A$ represent types of interactions with the environment. If m is the current state of the system and N is a measurable set of states, the function $\theta(\alpha)(m)$ is a measure on the state space and $\theta(\alpha)(m)(N) \in \mathbb{R}^+$ represents the *rate* of an exponentially distributed random variable that characterizes the duration of an α-transition from m to arbitrary $n \in N$. Indeterminacy in such systems is resolved by races between events executing at different rates.

Definition 3.1 (Markov kernels and Markov processes): Let (M, Σ) be an analytic space, where Σ is the Borel algebra generated by the topology, and A a denumerable set of labels. An *A-Markov kernel* is a tuple $\mathcal{M} = (M, \Sigma, \theta)$, with

$$\theta : A \to [\![M \to \Delta(M, \Sigma)]\!].$$

If $m \in M$ then the tuple (M, Σ, θ, m) is an *$\mathbb{A}$-Markov process* of $\mathcal{M}$ and m is its *initial state*.

Notice that $\theta(\alpha)$ is defined as a measurable mapping between (M, Σ) and the measurable space $\Delta(M, \Sigma)$ of the measures on (M, Σ). This condition is equivalent to the conditions on the two-variable *rate function* used in [14], [28] to define transitions for continuous Markov processes (see, e.g. Proposition 2.9, of [15]). If $\mathcal{M} = (M, \Sigma, \theta)$, we sometime denote the process (M, Σ, θ, m) by $(\mathcal{M}, m)$.

We define the stochastic bisimulation relation on MPs following the similar definitions of [12], [14], [28].

Given a binary relation $\mathfrak{R} \subseteq M \times M$ on a set M, we call a subset $N \subseteq M$ $\mathfrak{R}$-closed iff

$$\{m \in M \mid \exists n \in N, (n, m) \in \mathfrak{R}\} \subseteq N.$$

If (M, Σ) is a measurable space and $\mathfrak{R} \subseteq M \times M$, $\Sigma(\mathfrak{R})$ denotes the set of measurable $\mathfrak{R}$-closed subsets of M.

Definition 3.2 (Stochastic bisimulation): For an A-Markov kernel $\mathcal{M} = (M, \Sigma, \theta)$, a *rate-bisimulation relation* is an equivalence relation $\mathfrak{R} \subseteq M \times M$ such that $(m, n) \in \mathfrak{R}$ iff for any $C \in \Sigma(\mathfrak{R})$ and any $\alpha \in A$,

$$\theta(\alpha)(m)(C) = \theta(\alpha)(n)(C).$$

Two MPs $(\mathcal{M}, m)$ and $(\mathcal{M}, n)$ are *stochastic bisimilar*, written $m \sim_{\mathcal{M}} n$, if m and n are related by a rate-bisimulation relation.

Observe that, for any A-MP (M, Σ, θ), there exist rate-bisimulation relations. For instance, the identity of the elements of M is a rate-bisimulation relation.

IV. A minimal Stochastic Process Algebra

In this section we introduce a stochastic extension of CCS without replication [26]. As usual in stochastic process algebras, each transition a has associated a *rate* in $\mathbb{R}^+$ representing the absolute value of the parameter of an exponentially distributed random variable that characterizes the duration of an a-action. In addition, we also consider synchronizations of actions. As in CCS, the set of actions is equipped with an *involution* that associates to each action a its paired action $\overline{a}$; the paired actions have the same rates. The synchronization of $(a, \overline{a})$ counts as an *internal τ-action* with the rate satisfying the *mass action law* [6].

Formally, the set of *labels (actions)* is a countable set $\mathbb{A}$ endowed with (i) an *involution*, that is a function associating to each $a \in \mathbb{A}$ an element $\overline{a} \in \mathbb{A}$ such that $a \neq \overline{a}$ and $\overline{\overline{a}} = a$; (ii) a *weight function* $\iota : \mathbb{A} \to \mathbb{Q}^+$, such that for any $a \in \mathbb{A}$, $\iota(a) = \iota(\overline{a})$. In what follows we use two extensions of $\mathbb{A}$ defined for the *internal action* $\tau \notin \mathbb{A}$. On the syntactic level we involve the set $\mathbb{A}^* = \mathbb{A} \cup \{\tau_r \mid r \in \mathbb{Q}^+\}$, where indexed internal actions will be used for modelling delays in a system (the indexes represent the rates of the delays); we extend ι to $\mathbb{A}^*$ by $\iota(\tau_r) = r$. For the operational semantics we use the set of labels $\mathbb{A}^+ = \mathbb{A} \cup \{\tau\}$.

In what follows a, a', a_i denote arbitrary elements of $\mathbb{A}$, $\varepsilon, \varepsilon', \varepsilon_i$ denote arbitrary elements of $\mathbb{A}^*$ and $\alpha, \alpha', \alpha_i$ denote arbitrary elements of $\mathbb{A}^+$.

Definition 4.1 (Stochastic Processes): $\mathbb{A}$*-stochastic processes* are defined, on top of a constant 0 and for arbitrary $\varepsilon \in \mathbb{A}^*$, inductively as follows[1]

$$P := 0 \,\vdots\, \varepsilon.P \,\vdots\, P|P \,\vdots\, P + P.$$

We denote by $\mathbb{P}$ the set of stochastic processes.

An essential notion for processes is the *structural congruence relation* which equates processes that, in spite of their different syntactic form, represent the same systems.

Definition 4.2 (Structural congruence): Structural congruence is the smallest relation $\equiv \subseteq \mathbb{P} \times \mathbb{P}$ satisfying, for arbitrary $P, Q, R \in \mathbb{P}$ and $\varepsilon \in \mathbb{A}^*$ the following conditions.

I. $\equiv$ is an equivalence relation on $\mathbb{P}$

II. $(\mathbb{P}, |, 0)$ is a commutative monoid for $\equiv$, i.e.,
1. $P|Q \equiv Q|P$; 2. $(P|Q)|R \equiv P|(Q|R)$; 3. $P|0 \equiv P$.

III. $(\mathbb{P}, +, 0)$ is a commutative monoid for $\equiv$, i.e.,
1. $P + Q \equiv Q + P$; 2. $(P + Q) + R \equiv P + (Q + R)$; 3. $P + 0 \equiv P$.

IV. $\equiv$ is a congruence with respect to the algebraic structure of $\mathbb{P}$, i.e., if $P \equiv Q$, then
1. $P|R \equiv Q|R$; 2. $P + R \equiv Q + R$; 3. $\varepsilon.P \equiv \varepsilon.Q$.

Let $\mathbb{P}^{\equiv}$ be the set of $\equiv$-equivalence classes on $\mathbb{P}$. For arbitrary $P \in \mathbb{P}$, we denote by $P^{\equiv}$ the $\equiv$-equivalence class of P. Note that $\mathbb{P}^{\equiv}$ is a denumerable partition of $\mathbb{P}$, hence it generates a σ-algebra Π over $\mathbb{P}$; thus, $(\mathbb{P}, \Pi)$ is a measurable space. The measurable sets are (possibly denumerable) reunions of $\equiv$-equivalence classes on $\mathbb{P}$. In what follows we use $\mathcal{P}, \mathcal{P}_i, \mathcal{R}, \mathcal{Q}$ to denote arbitrary measurable sets of Π.

For the economy of the paper it is useful to define the following operations on the sets of Π. For arbitrary $\mathcal{P}, \mathcal{Q} \in \Pi$ and $P \in \mathbb{P}$, consider

$$\mathcal{P}|\mathcal{Q} = \bigcup_{P \in \mathcal{P}, Q \in \mathcal{Q}} (P|Q)^{\equiv} \quad \text{and} \quad \mathcal{P}_P = \bigcup_{P|R \in \mathcal{P}} R^{\equiv}.$$

Notice that $\mathcal{P}|\mathcal{Q}$ and $\mathcal{P}_P$ are measurable sets.

In the rest of this section we show that the measurable space $(\mathbb{P}, \Pi)$ of stochastic processes can be organised as an $\mathbb{A}^+$-Markov kernel. This will implicitly provide a structural operational semantics for the minimal stochastic process algebra such that the behavioural equivalence coincides with the bisimulation of MPs.

Notice, to begin with, that $(\mathbb{P}, \Pi)$ is a Polish, hence, analytic space.

The next definition constructs, inductively on the structure of processes, a function $\theta : \mathbb{A}^+ \to [\![\mathbb{P} \to \Delta(\mathbb{P}, \Pi)]\!]$ which, we will prove, organizes $(\mathbb{P}, \Pi, \theta)$ as an $\mathbb{A}^+$-Markov kernel. The intuition is that for arbitrary $P \in \mathbb{P}$, $\mathcal{P} \in \Pi$ and $\alpha \in \mathbb{A}^+$, $\theta(\alpha)(P)(\mathcal{P})$ represents the total rate of the α actions from P to (elements of) $\mathcal{P}$.

Recall that $\omega, D(r, Q^{\equiv}) \in \Delta(\mathbb{P}, \Pi)$ denote the null measure and the r-Dirac measure[2] on $Q^{\equiv}$ respectively.

Definition 4.3: Let $\theta : \mathbb{A}^+ \to [\mathbb{P} \to \Delta(\mathbb{P}, \Pi)]$ be defined, inductively on the structure of $P \in \mathbb{P}$, as follows.

The case $P = 0$: For any $\alpha \in \mathbb{A}^+$, let $\theta(\alpha)(0) = \omega$.

The case $P = \varepsilon.Q$, $\varepsilon \in \mathbb{A}^*$: For arbitrary $a \in \mathbb{A}$, let

$$\theta(\tau)(\varepsilon.Q) = \begin{cases} D(\iota(\varepsilon), P^{\equiv}), & \varepsilon \notin \mathbb{A} \\ \omega, & \varepsilon \in \mathbb{A} \end{cases}$$

$$\theta(a)(\varepsilon.Q) = \begin{cases} D(\iota(\varepsilon), P^{\equiv}), & \varepsilon = a \\ \omega, & \varepsilon \neq a \end{cases}$$

The case $P = Q + R$. For any $\alpha \in \mathbb{A}^+$ and $\mathcal{P} \in \Pi$,

$$\theta(\alpha)(Q + R)(\mathcal{P}) = \theta(\alpha)(Q)(\mathcal{P}) + \theta(\alpha)(R)(\mathcal{P}).$$

The case $P = Q|R$. For any $a \in \mathbb{A}$ and $\mathcal{P} \in \Pi$,

$$\theta(a)(Q|R)(\mathcal{P}) = \theta(a)(R)(\mathcal{P}_Q) + \theta(a)(Q)(\mathcal{P}_R),$$

$$\theta(\tau)(Q|R)(\mathcal{P}) = \theta(\tau)(R)(\mathcal{P}_Q) + \theta(\tau)(Q)(\mathcal{P}_R) +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{P}}^{a \in \mathbb{A}} \frac{\theta(a)(Q)(\mathcal{P}_1) \cdot \theta(\overline{a})(R)(\mathcal{P}_2)}{2 \cdot \iota(a)}.$$

If we define the *set of active actions of a process* $P \in \mathbb{P}$ by $act(0) = \emptyset$, $act(a.P) = \{a\}$, $act(P + Q) = act(P|Q) = act(P) \cup act(Q)$, then any process has only a finite set of active actions. Notice that $\theta(a)(P) \neq \omega$ iff $a \in act(P)$. This means that for any $a \notin act(P)$ and any $\mathcal{R} \in \Pi$, $\theta(a)(P)(\mathcal{R}) = 0$. Consequently, the infinitary sum involved in Definition 4.3 has a finite number of non-zero summands. Notice also that the sum is divided by 2 because we count the interaction pairs $(a, \overline{a})$ twice (recall that $a = \overline{\overline{a}}$) and is divided by $\iota(a)$ to satisfy the mass action law[3].

The next theorem states that the space of processes with the function defined above is an $\mathbb{A}^+$-Markov kernel. Notice that, for proving this result, we implicitly show the correctness of the previous definition, i.e. that for each $\alpha \in \mathbb{A}^+$ and each $P \in \mathbb{P}$, $\theta(\alpha)(P) \in \Delta(\mathbb{P}, \Pi)$, i.e. it is a measure. From here it follows immediately that for each $\alpha \in \mathbb{A}^+$, $\theta(\alpha) \in [\![\mathbb{P} \to \Delta(\mathbb{P}, \Pi)]\!]$.

Theorem 4.1: $(\mathbb{P}, \Pi, \theta)$ is an $\mathbb{A}^+$-Markov kernel.

A consequence of the previous theorem is that for each $P \in \mathbb{P}$, $(\mathbb{P}, \Pi, \theta, P)$ is a Markov process. In effect, we can define a stochastic bisimulation for the elements of our process algebra simply as stochastic bisimulation of Markov processes in $(\mathbb{P}, \Pi, \theta)$.

[1] Notice that in practice we cannot measure nor specify models with irrational rates and for this reason we have chosen $\iota(\varepsilon) \in \mathbb{Q}^+$ for all $\varepsilon \in \mathbb{A}^*$.

[2] Notice that $\{Q^{\equiv}, Q \in \mathbb{P}\}$ is a base of Π, hence, for any $r \in \mathbb{R}^+$ we can define the r-Dirac measure $D(r, Q^{\equiv})$ on arbitrary $Q^{\equiv}$.

[3] Recall that $\iota(a) = \iota(\overline{a})$ and this value is involved both in $\theta(a)(Q)(\mathcal{P}_1)$ and in $\theta(\overline{a})(R)(\mathcal{P}_2)$.

V. Structural Operational Semantics

In this section we introduce the structural operational semantics for the minimal process algebra, with the intention to induce a behavioural equivalence on processes that coincides with their bisimulation as MPs. In this case we do not associate to each tuple *(process, action, process)* a rate, as usual in stochastic process algebras, because a transition in our case is not between two processes, but from a process to an infinite measurable set of processes. However, our intention is to maintain "the spirit" of process algebras and for this reason we will replace the classic rules based on transitions of type $P \to Q$ with rules based on "generalised" transitions of type $P \to \mu$ where $\mu : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$ is a function defining a class of $\mathbb{A}^+$-indexed measures on $(\mathbb{P}, \Pi)$.

For simplifying the rules of the operational semantics, we first define some operations on the functions in $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ and analyze their mathematical structures and properties.

Recall that ω is the null measure and $D(r, P^{\equiv})$ is the r-Dirac measure on $P^{\equiv}$. We say that a function $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ has *finite support* if $\mathbb{A} \setminus \mu^{-1}(\omega)$ is finite or empty.

Definition 5.1: Consider the following constants and operations on $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ defined as follows.

1. For arbitrary $\alpha \in \mathbb{A}^+$, let $\overline{\omega} : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$ be defined by $\overline{\omega}(\alpha) = \omega$.

2. For arbitrary $\varepsilon \in \mathbb{A}^*$ and $P \in \mathbb{P}$ let $[^{\varepsilon}_{P}] : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$ be defined, for arbitrary $a \in \mathbb{A}$, by

$$[^{\varepsilon}_{P}](a) = \begin{cases} D(\iota(\varepsilon), P^{\equiv}), & a = \varepsilon \\ \omega, & a \neq \varepsilon \end{cases}$$

$$[^{\varepsilon}_{P}](\tau) = \begin{cases} D(\iota(\varepsilon), P^{\equiv}), & \varepsilon \notin \mathbb{A} \\ \omega, & \varepsilon \in \mathbb{A} \end{cases}$$

3. For arbitrary $\mu', \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$, let $\mu' \oplus \mu'' : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$ be defined, for arbitrary $\alpha \in \mathbb{A}^+$, by

$$(\mu' \oplus \mu'')(\alpha) = \mu'(\alpha) + \mu''(\alpha).$$

4. For arbitrary $\mu', \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ with finite support and $P, Q \in \mathbb{P}$, let $\mu' \ {}_{P}{\otimes}_{Q}\ \mu'' : \mathbb{A}^+ \to \Delta(\mathbb{P}, \Pi)$ be defined by,

$$(\mu' \ {}_{P}{\otimes}_{Q}\ \mu'')(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}_Q) + \mu''(a)(\mathcal{R}_P) \text{ for } a \in \mathbb{A}$$

$$(\mu' \ {}_{P}{\otimes}_{Q}\ \mu'')(\tau)(\mathcal{R}) = \mu'(\tau)(\mathcal{R}_Q) + \mu''(\tau)(\mathcal{R}_P) + \sum_{\mathcal{P}_1|\mathcal{P}_2 \subseteq \mathcal{R}}^{a \in \mathbb{A}} \frac{\mu'(a)(\mathcal{P}_1) \cdot \mu''(\overline{a})(\mathcal{P}_2)}{2 \cdot \iota(a)}.$$

Observe that because μ' and μ'' have finite support, the sum involved in the definition of ${}_{P}{\otimes}_{Q}$ has a finite number of non-zero summands.

The next lemma proves that the definitions of $\oplus$ and ${}_{P}{\otimes}_{Q}$ for arbitrary $P, Q \in \mathbb{P}$ are correct; it also states some basic properties of these operators.

Lemma 5.1: 1. For arbitrary $\mu, \mu', \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$, $\mu \oplus \mu' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ and
(a). $\mu \oplus \mu' = \mu' \oplus \mu$,
(b). $(\mu \oplus \mu') \oplus \mu'' = \mu \oplus (\mu' \oplus \mu'')$,
(c). $\mu = \mu \oplus \overline{\omega}$.
2. For arbitrary $P, Q, R \in \mathbb{P}$ and $\mu', \mu'', \mu''' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ with finite support, $\mu \ {}_{P}{\otimes}_{Q}\ \mu' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ and
(a). $\mu' \ {}_{P}{\otimes}_{Q}\ \mu'' = \mu'' \ {}_{Q}{\otimes}_{P}\ \mu'$,
(b). $(\mu' \ {}_{P}{\otimes}_{Q}\ \mu'') \ {}_{P|Q}{\otimes}_{R}\ \mu''' = \mu' \ {}_{P}{\otimes}_{Q|R}\ (\mu'' \ {}_{Q}{\otimes}_{R}\ \mu''')$,
(c). $\mu' \ {}_{P}{\otimes}_{0}\ \overline{\omega} = \mu'$.
3. For arbitrary $P, P', Q, Q' \in \mathbb{P}$, $\varepsilon \in \mathbb{A}^+$ and $\mu', \mu'' \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ with finite support,
(a). if $P \equiv P'$ and $Q \equiv Q'$, then $\mu' \ {}_{P}{\otimes}_{Q}\ \mu'' = \mu' \ {}_{P'}{\otimes}_{Q'}\ \mu''$,
(b). if $P \equiv Q$, then $[^{\varepsilon}_{P}] = [^{\varepsilon}_{Q}]$.

The rules of the structural operational semantics, given for arbitrary $P, Q \in \mathbb{P}$ and $\varepsilon \in \mathbb{A}^+$, are listed in Table I. The *stochastic transition relation* is the smallest relation $\to \subseteq \mathbb{P} \times \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ satisfying these rules.

$$(Null).\qquad \frac{}{0 \to \overline{\omega}}$$

$$(Guard).\qquad \frac{}{\varepsilon.P \to [^{\varepsilon}_{P}]}$$

$$(Sum).\qquad \frac{P \to \mu' \qquad Q \to \mu''}{P + Q \to \mu' \oplus \mu''}$$

$$(Par).\qquad \frac{P \to \mu' \qquad Q \to \mu''}{P|Q \to \mu' \ {}_{P}{\otimes}_{Q}\ \mu''}$$

TABLE I
Structural Operational Semantics

The operational semantics associates to each process $P \in \mathbb{P}$ a mapping $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$. For each $\equiv$-closed set of processes $\mathcal{P} \in \Pi$ and each $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(\mathcal{P}) \in \mathbb{R}^+$ represents the total rate of the α-reductions of P to some arbitrary element of $\mathcal{P}$. The next lemma guarantees the consistency of the relation $\to$ and of our operational semantics.

Lemma 5.2: For any $P \in \mathbb{P}$ there exists a unique $\mu \in \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ such that $P \to \mu$; moreover, μ has finite support.

The operational semantics can be further used to define various pointwise semantics as, for instance, by $P \xrightarrow{\alpha, s} Q$ iff $\mu(\alpha)(Q^{\equiv}) = s$.

Example 5.1: Suppose that $a, b, c \in \mathbb{A}$ with $a, \overline{a}, b, \overline{b}, c, \overline{c}$ pairwise distinct and $\iota(a) = r$. It is immediate that

1. $a.P|a.P \xrightarrow{a, 2r} a.P|P$, 2. $a.P|\overline{a}.Q \xrightarrow{\tau_r, r} P|Q$,
3. $(a.P_1 + b.P_2)|(\overline{a}.Q_1 + c.Q_2) \xrightarrow{\tau_r, r} P_1|Q_1$.

The next lemma ensures that the operational semantics does not differentiate the structural congruent processes.

Lemma 5.3: If $P \equiv Q$ and $P \to \mu$, then $Q \to \mu$.

In general, we can speak of lifting the algebraic structure of the class $\mathbb{P}$ of processes to the class $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ of functions. However, it is not an "authentic" lifting as the signature on $\mathbb{P}$ is not the same as the signature of $\Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ – a fact that differentiates our approach from the other GSOS [32] or SGSOS [22] formats. For instance, to the parallel operator "|" there corresponds, in the domain of functions, a denumerable class of binary operators indexed by processes, i.e. "${}_{P}{\otimes}_{Q}$". This non-standard situation is a consequence of the fact that $\equiv \subsetneq \approx$. If we consider the processes $P = a.0|b.0$ and $Q =$

$a.b.0 + b.a.0$ for $a, b \in \mathbb{A}$ and $\{a, \overline{a}\} \cap \{b, \overline{b}\} = \emptyset$, then $P \to (\mu_1 = [^a_0]\ {}_{a.0}\!\otimes_{b.0}\ [^b_0])$ and $Q \to (\mu_2 = [^{\ a}_{b.0}] \oplus [^{\ b}_{a.0}])$. It is trivial to verify that $\mu_1 = \mu_2$, however $P \not\equiv Q$. This shows that for some $R \to \nu$, we can have $\mu_1\ {}_P\!\otimes_R \nu \neq \mu_1\ {}_Q\!\otimes_R \nu$. Hence, due to the parallel operator and to the σ-algebra we have chosen, it is not possible to provide an SOS that uses the same signature for processes and for behaviours (functions) as in the classic case. However, in Section VI, after introducing the stochastic bisimulation relation "$\sim$" for processes and functions, we will see that the quotient of $\sim$ on both domains produces the same algebraic signatures, meaning that we eventually have a "well-behaved" SOS, but up to stochastic bisimulation.

It is also important to notice that our "generalised" transition system induced by our SOS is image-finite. The importance of this property was motivated from the perspective of GSOS. In [32] it is observed that image-finite GSOS are in one-to-one correspondence with the *distributive laws* that ensure the cooperation between the algebraic and the coalgebraic structures of the class of processes, and this was eventually proved in [1]. The next lemma shows that our system has a similar property. We write $P \Longrightarrow Q$ if there exists $\alpha \in \mathbb{A}^+$ and $r \neq 0$ such that $P \xrightarrow{\alpha,r} Q^{\equiv}$ and let $\Longrightarrow^*$ be the transitive closure of $\Longrightarrow$.

Lemma 5.4: For an arbitrary process $P \in \mathbb{P}$, the sets $\{\alpha \in \mathbb{A}^+ \mid P \xrightarrow{\alpha,r} \mathbb{P}, r \neq 0\}$, $\{Q^{\equiv} \in \Pi \mid P \Longrightarrow Q\}$ and $\{Q^{\equiv} \in \Pi \mid P \Longrightarrow^* Q\}$ are finite.

Of particular importance for the metrics we will introduce later are the second and the third sets of the previous lemma that allow us to give inductive definitions on the generalised transition tree.

VI. Stochastic bisimulation is a congruence

This section is dedicated to the study of stochastic bisimulation for the minimal stochastic process algebra. In the pointwise approach, since the operational semantics requires various mathematical artifacts such as the multi-transition systems [18], [19] or the proved SOS [29], the problem of stochastic bisimulation is difficult to trace. Recently, an elegant solution was proposed in [22] for the case when there are no equational restrictions on the algebraic level. As argued before, for practical modeling purposes, our algebra is endowed with an equational theory of structural congruence that organizes the measurable space of processes and consequently, stochastic bisimulation requires a different treatment.

In what follows we introduce the stochastic bisimulation for the minimal process algebra as the stochastic bisimulation on the Markov kernel $(\mathbb{P}, \Pi, \theta)$. We show that it behaves well both on coalgebraic and on algebraic levels: processes that have associated the same functions by our SOS are bisimilar and the bisimulation is a congruence that extends the structural congruence.

Before proceeding with the technical developments we informally recall that, in abstract algebra, given a set $\mathcal{X}$ with an algebraic structure, a *congruence relation* on $\mathcal{X}$ is an equivalence relation on $\mathcal{X}$ preserving its algebraic structure.

Lemma 5.2 shows that the operational semantics induces a function $\vartheta : \mathbb{P} \to \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ defined by

$$\vartheta(P) = \mu \text{ iff } P \to \mu.$$

In this light, one can note a relation between ϑ and the function θ that organises $\mathbb{P}$ as a Markov kernel. It reflects the similarity between the Definitions 4.3 and 5.1.

Lemma 6.1: If $(\mathbb{P}, \Pi, \theta)$ is the Markov kernel of processes and $\vartheta : \mathbb{P} \to \Delta(\mathbb{P}, \Pi)^{\mathbb{A}^+}$ is the function induced by SOS, then for any $P \in \mathbb{P}$, $\alpha \in \mathbb{A}^+$ and $\mathcal{P} \in \Pi$,

$$\theta(\alpha)(P)(\mathcal{P}) = \vartheta(P)(\alpha)(\mathcal{P}).$$

Recall that for a Markov kernel (M, Σ, θ), $\sim_{(M,\Sigma,\theta)}$ denotes the stochastic bisimulation on it. The next result is a direct consequence of the previous lemma stating that $\sim_{(\mathbb{P},\Pi,\theta)}$ is an extension of the kernel of ϑ.

Corollary 6.1: For arbitrary $P, Q \in \mathbb{P}$, if $P \to \mu$ and $Q \to \mu$, then $P \sim_{(\mathbb{P},\Pi,\theta)} Q$.

This result guarantees that we can safely define the stochastic bisimulation for our process algebra as the stochastic bisimulation on $(\mathbb{P}, \Pi, \theta)$. This allows us to propose the next definition.

Definition 6.1 (Stochastic bisimulation on processes): A *rate-bisimulation relation* on processes is an equivalence relation $\mathfrak{R} \subseteq \mathbb{P} \times \mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$ with $P \to \mu$ and $Q \to \mu'$, $(P, Q) \in \mathfrak{R}$ iff for any $C \in \Pi(\mathfrak{R})$ and any $\alpha \in \mathbb{A}^+$,

$$\mu(\alpha)(C) = \mu'(\alpha)(C).$$

Two processes $P, Q \in \mathcal{P}$ are stochastic bisimilar, written $P \sim Q$, iff there exists a rate bisimulation relation $\mathfrak{R}$ such that $(P, Q) \in \mathfrak{R}$.

The next theorem provides a characterization of stochastic bisimulation.

Theorem 6.1: The stochastic bisimulation $\sim$ is the smallest equivalence relation on $\mathbb{P}$ such that for arbitrary $P, Q \in \mathbb{P}$ with $P \to \mu$ and $Q \to \mu'$, $P \sim Q$ iff for any $C \in \Pi(\sim)$ and any $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C) = \mu'(\alpha)(C)$.

We denote by $\mathbb{P}^{\sim}$ the set of $\sim$-equivalence classes on $\mathbb{P}$, and for arbitrary $P \in \mathbb{P}$ we denote by $P^{\sim}$ the $\sim$-equivalence class containing P. We use the notation $P \not\sim Q$ to say that P and Q are not stochastic bisimilar.

In what follows we show some examples of bisimilar processes. The first example proves a general rule for concurrent Markovian processes (see Section 4.1.2 of [21]).

Example 6.1: (i) If $a, b \in A$ such that $\overline{a} \neq b$, then for any $P, Q \in \mathbb{P}$, $a.P|b.Q \sim a.(P|b.Q) + b.(a.P|Q)$.
Observe that $a.P|b.Q \to [^a_P]\ {}_{a.P}\!\otimes_{b.Q}\ [^b_Q]$ and $a.(P|b.Q) + b.(a.P|Q) \to [^{\ \ a}_{P|b.Q}] \oplus [^{\ \ b}_{a.P|Q}]$. It is simple to verify that, for arbitrary $C \in \mathbb{P}^{\sim}$, we have

$$[^a_P]\ {}_{a.P}\!\otimes_{b.Q}\ [^b_Q](x)(C) = [^{\ \ a}_{P|b.Q}] \oplus [^{\ \ b}_{a.P|Q}](x)(C) =$$

$$\begin{cases} \iota(a) & \text{if } x = a,\ P|b.Q \in C, \\ 0 & \text{if } x = a,\ P|b.Q \notin C, \\ \iota(b) & \text{if } x = b,\ a.P|Q \in C, \\ 0 & \text{if } x = b,\ a.P|Q \notin C, \\ 0 & \text{if } x \notin \{a, b\}. \end{cases}$$

Hence, $a.P|b.Q \sim a.(P|b.Q) + b.(a.P|Q)$.

(ii) $\tau_r.P|\tau_s.Q \sim \tau_r.(P|\tau_s.Q) + \tau_s.(\tau_r.P|Q)$.

As before, we have $\tau_r.P|\tau_s.Q \rightarrow [{}^{\tau_r}_{P}]\ {}_{a.P}\!\otimes_{b.Q} [{}^{\tau_s}_{Q}]$ and $\tau_r.(P|\tau_s.Q)+\tau_s.(\tau_r.P|Q) \rightarrow [{}^{\tau_r}_{P|\tau_s.Q}] \oplus [{}^{\tau_s}_{a.P|Q}]$. One can verify that

$$[{}^{\tau_r}_{P}]\ {}_{\tau_r.P}\!\otimes_{\tau_s.Q} [{}^{\tau_s}_{Q}](x)(C) = [{}^{\tau_r}_{P|\tau_s.Q}] \oplus [{}^{\tau_s}_{a.P|Q}](x)(C) =$$

$$\begin{cases} r & \text{if } x=\tau,\ P|\tau_s.Q \not\sim \tau_r.P|Q,\ P|\tau_s.Q \in C, \\ 0 & \text{if } x=\tau,\ P|\tau_s.Q \not\sim \tau_r.P|Q,\ P|\tau_s.Q \notin C, \\ s & \text{if } x=\tau,\ P|\tau_s.Q \not\sim \tau_r.P|Q,\ \tau_r.P|Q \in C, \\ 0 & \text{if } x=\tau,\ P|\tau_s.Q \not\sim \tau_r.P|Q,\ \tau_r.P|Q \notin C, \\ r+s & \text{if } x=\tau,\ P|\tau_s.Q \sim \tau_r.P|Q \in C, \\ 0 & \text{if } x=\tau,\ P|\tau_s.Q \sim \tau_r.P|Q \notin C, \\ 0 & \text{if } x \neq \tau. \end{cases}$$

Example 6.2: Let $b, c \in \mathbb{A}$ be such that $\bar{b} \neq c$. In Example 6.1 we have seen that $b.0|c.0 \sim b.c.0+c.b.0$. Consider the processes $P = \tau_r.(b.0|c.0)+\tau_r.(b.c.0+c.b.0)$, $Q = \tau_r.(b.0|c.0)+\tau_r.(b.0|c.0)$ and $R = \tau_r.(b.c.0+c.b.0)+\tau_r.(b.c.0+c.b.0)$.

If C is the $\sim$-equivalence class that contains $b.0|c.0$ and $b.c.0+c.b.0$, then $P \xrightarrow{\tau,2r} C$, $Q \xrightarrow{\tau,2r} C$, $R \xrightarrow{\tau,2r} C$ and for any other $\sim$-equivalence class C', $P \xrightarrow{\tau,0} C'$, $Q \xrightarrow{\tau,0} C'$ and $R \xrightarrow{\tau,0} C'$. Consequently, $P \sim Q \sim R$ (also because for any other action the rate is 0 everywhere). On the other hand, if we consider instead the pointwise semantics, then we obtain

$$P \xrightarrow{\tau,r} b.0|c.0 \text{ and } P \xrightarrow{\tau,r} b.c.0+c.b.0,$$

$$Q \xrightarrow{\tau,2r} b.0|c.0 \text{ and } Q \xrightarrow{\tau,0} b.c.0+c.b.0,$$

$$R \xrightarrow{\tau,0} b.0|c.0 \text{ and } R \xrightarrow{\tau,2r} b.c.0+c.b.0.$$

Notice that, in spite of the fact that the three processes are bisimilar, they are not agreeing on any "pointwise" transition. This emphasizes the difficulties risen by the pointwise semantics for the case of stochastic process algebras.

The relation $\sim$ on $\mathbb{P}$ can be lifted to $\Delta(P)^{\mathbb{A}^+}$ by defining, for arbitrary $\mu, \mu' \in \Delta(\mathbb{P})^{\mathbb{A}^+}$, $\mu \sim \mu'$ iff for any $C \in \mathbb{P}^{\sim}$ and any $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C) = \mu'(\alpha)(C)$. Notice that $\sim \subseteq \Delta(\mathbb{P})^{\mathbb{A}^+} \times \Delta(\mathbb{P})^{\mathbb{A}^+}$ is an equivalence relation. We denote by $(\Delta(\mathbb{P})^{\mathbb{A}^+})^{\sim}$ the set of $\sim$-equivalence classes on $\Delta(\mathbb{P})^{\mathbb{A}^+}$ and for an arbitrary $\mu \in \Delta(\mathbb{P})^{\mathbb{A}^+}$ we denote by $\mu^{\sim}$ the $\sim$-equivalence class of μ.

With this notation, from Theorem 6.1 we derive the next corollary.

Corollary 6.2: Given $P, Q \in \mathbb{P}$, if $P \rightarrow \mu$ and $Q \rightarrow \mu'$, then $P \sim Q$ iff $\mu \sim \mu'$.

A consequence of $\sim$ being an equivalence on $\Delta(\mathbb{P})^{\mathbb{A}^+}$ is the next theorem that shows that our processes behave "correctly" with respect to structural congruence.

Theorem 6.2: Given $P, Q \in \mathbb{P}$, if $P \equiv Q$, then $P \sim Q$.

In addition to the result of the previous theorem, notice that $\sim$ is strictly larger than $\equiv$, because for arbitrary $a, b \in \mathbb{A}$ we have $a.0|b.0 \sim a.b.0+b.a.0$ and $a.0|b.0 \not\equiv a.b.0+b.a.0$.

We now state the main theorem of this section.

Theorem 6.3 (Congruence): Stochastic bisimulation on $\mathbb{P}$ is a congruence relation with respect to the algebraic structure of $\mathbb{P}$, i.e. for arbitrary $P, P', Q, Q' \in \mathbb{P}$ and $\varepsilon \in \mathbb{A}^*$,
1. if $P \sim P'$, then $\varepsilon.P \sim \varepsilon.P'$;
2. if $P \sim P'$ and $Q \sim Q'$, then $P+Q \sim P'+Q'$;
3. if $P \sim P'$ and $Q \sim Q'$, then $P|Q \sim P'|Q'$.

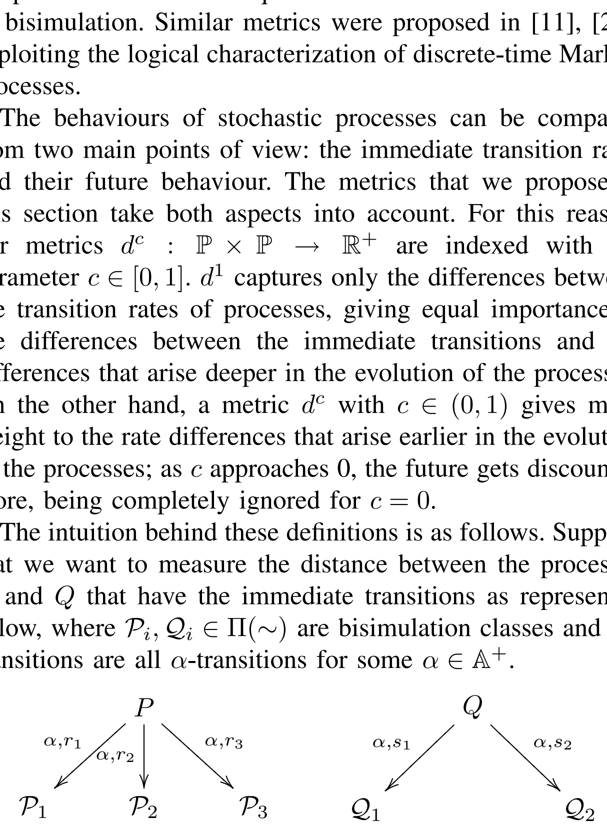
Because $ker(\vartheta) \subseteq \sim$ and $\sim$ is a congruence for processes, we deduce that if $P \sim P'$, $Q \sim Q'$, $P \rightarrow \mu$, $P' \rightarrow \mu'$, $Q \rightarrow \nu$ and $Q' \rightarrow \nu'$, then $\mu\ {}_{P}\!\otimes_{Q} \nu \sim \mu'\ {}_{P}\!\otimes_{Q'} \nu'$ and for any $\varepsilon \in \mathbb{A}^*$, $[{}^{\varepsilon}_{P}] \sim [{}^{\varepsilon}_{P'}]$. This shows that by taking the quotient with $\sim$ both on processes and on functions, we will obtain identical signatures for processes and for behaviors (functions) and one could provide a SOS format in the stile of [32], [22].

VII. Metrics for stochastic processes

In the case of stochastic and probabilistic systems, bisimulation is a strict concept: it verifies whether two processes have identical behaviours. In applications we need more. For instance, we want to know whether two processes that may differ by only a small amount in real-valued parameters (rates or probabilities) are behaving in a similar way. To solve this problem we define some pseudometrics on the set of processes of the minimal process algebra that will measure how much two processes are alike in terms of behaviour. In this sense, two processes are at distance zero iff they are bisimilar. Thus, the pseudometrics will be quantitative extensions of the notion of bisimulation. Similar metrics were proposed in [11], [28], exploiting the logical characterization of discrete-time Markov processes.

The behaviours of stochastic processes can be compared from two main points of view: the immediate transition rates and their future behaviour. The metrics that we propose in this section take both aspects into account. For this reason, our metrics $d^c : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{R}^+$ are indexed with the parameter $c \in [0,1]$. d^1 captures only the differences between the transition rates of processes, giving equal importance to the differences between the immediate transitions and the differences that arise deeper in the evolution of the processes. On the other hand, a metric d^c with $c \in (0,1)$ gives more weight to the rate differences that arise earlier in the evolution of the processes; as c approaches 0, the future gets discounted more, being completely ignored for $c = 0$.

The intuition behind these definitions is as follows. Suppose that we want to measure the distance between the processes P and Q that have the immediate transitions as represented below, where $\mathcal{P}_i, \mathcal{Q}_i \in \Pi(\sim)$ are bisimulation classes and the transitions are all α-transitions for some $\alpha \in \mathbb{A}^+$.



For calculating the distance $d^c(P,Q)$, we first pair classes $\mathcal{P}_i$ and $\mathcal{Q}_j$ and then sum the differences between the rates of going from P and Q to $\mathcal{P}_i$ and $\mathcal{Q}_j$, respectively, and the weighted distance between arbitrary processes $P_i \in \mathcal{P}_i$ and

$Q_j \in \mathcal{Q}_j$. We thus obtain, for the pair $(\mathcal{P}_i, \mathcal{Q}_j)$, the value $|r_i - s_j| + c \cdot d^c(P_i, Q_j)$. There are various ways in which one can take these pairs: d^c is the infimum of the values one can get taking all possible pairings of bisimulation classes. However, these pairings have to be one-to-one and onto on $\mathbb{P}^\sim$ and for this reason we will use the possible bijections on $\mathbb{P}^\sim$. Another observation is that we only need to consider the pairs $(\mathcal{P}_i, \mathcal{Q}_j)$, such that either P can do an α-transition to $\mathcal{P}_i$ with non-zero rate, or Q can do an α-transition to $\mathcal{Q}_j$ with non-zero rate.

In what follows we formalize these intuitions. We introduce two families of metrics on $\mathbb{P}$, $\mathbb{D}_\alpha$ for $\alpha \in \mathbb{A}^+$ and $\mathbb{D}$. The first family contains measures that concern only α-transitions for a fixed α, while the second takes into account all the transitions.

As before, for arbitrary $P, Q \in \mathbb{P}$, we write $P \Longrightarrow Q$ if there exists $\alpha \in \mathbb{A}^+$ and $r \neq 0$ such that $P \xrightarrow{\alpha,r} Q^\equiv$. Let

$$\mathcal{D}(P) = \bigcup_{P \Longrightarrow Q} Q^\sim$$

called the set of derivatives of P. Let $\mathcal{B}$ be the set of bijections $\sigma : \mathbb{P}^\sim \to \mathbb{P}^\sim$. For arbitrary $P, Q, R, S \in \mathbb{P}$ and $\sigma \in \mathcal{B}$ we write $R[\sigma_Q^P]S$ if $R \in \mathcal{D}(P) \cup \sigma^{-1}(\mathcal{D}(Q))$ and $S^\sim = \sigma(R^\sim)$.

Definition 7.1: For arbitrary $\alpha \in \mathbb{A}^+$ consider the family $\mathbb{D}_\alpha$ of functions $d_\alpha^c : \mathbb{P} \times \mathbb{P} \to \mathbb{R}^+$, $c \in [0,1]$, defined by
(i) $d_\alpha^c(0,0) = 0$;
(ii) for $P', P'' \in \mathbb{P}$ with $P' \to \mu'$ and $P'' \to \mu''$,
$d_\alpha^c(P', P'') = \inf_{\sigma \in \mathcal{B}} \sigma_\alpha(P', P'')$, where[4]

$$\sigma_\alpha(P', P'') = \sum_{(R^\sim, S^\sim)}^{R[\sigma_{P''}^{P'}]S} (|\mu'(\alpha)(R^\sim) - \mu''(\alpha)(S^\sim)| + c \cdot d_\alpha^c(R, S)).$$

The correctness of this definition derives from Lemma 5.4 and from the fact that the transition tree of a derivative of a process P is strictly less complex than the transition tree of P. The same arguments guarantee that the supremum considered before is over a set with a finite number of non-zero elements.

The parameter $c \in [0,1]$ is used to associate a weight to each transition step. For instance if $a \in \mathbb{A}$, then $d_\tau^c(\tau_1.0, \tau_1.0 + \tau_1.0) = |2-1| = 1$ because the first is doing a τ-transition with rate 1 and the second with rate 2, $d_\tau^c(\tau_1.0, \tau_1.0 + a.0) = |1-1| = 0$ because both are doing τ-transitions with rate 1 and for similar reasons $d_\tau^c(\tau_1.0 + \tau_1.0, \tau_1.0 + a.0) = |2-1| = 1$.

$$\begin{tikzcd}
\tau_1.0 \arrow[d, "{\tau,1}"'] & \tau_1.0+\tau_1.0 \arrow[d, "{\tau,2}"'] & \tau_1.0+a.0 \arrow[d, "{\tau,1}"'] \arrow[dr, "{a,\iota(a)}"] & \\
0^\sim & 0^\sim & 0^\sim & 0^\sim
\end{tikzcd}$$

If we prefix these three processes, we will see their difference only at the second level transitions and this will influence the measure. Thus, $d_\tau^c(\tau_2.\tau_1.0, \tau_2.(\tau_1.0+\tau_1.0)) = |2-2| + c \cdot |2-1| = c$, $d_\tau^c(\tau_2.\tau_1.0, \tau_2.(\tau_1.0+a.0)) = |2-2| + c \cdot |1-1| = 0$ and $d_\tau^c(\tau_2.(\tau_1.0+\tau_1.0), \tau_2.(\tau_1.0+a.0)) = |2-2| + c \cdot |2-1| = c$.

[4]The sum is for all pairs $(R^\sim, S^\sim)$ such that $R[\sigma_{P''}^{P'}]S$.

$$\begin{tikzcd}
\tau_2.\tau_1.0 \arrow[d, "{\tau,2}"'] & \tau_2.(\tau_1.0+\tau_1.0) \arrow[d, "{\tau,2}"'] & \tau_2.(\tau_1.0+a.0) \arrow[d, "{\tau,2}"'] & \\
\tau_1.0^\sim \arrow[d, "{\tau,1}"'] & \tau_1.0+\tau_1.0^\sim \arrow[d, "{\tau,2}"'] & \tau_1.0+a.0^\sim \arrow[d, "{\tau,1}"'] \arrow[dr, "{a,1}"] & \\
0^\sim & 0^\sim & 0^\sim & 0^\sim
\end{tikzcd}$$

We can use various values of $c \in [0,1]$ to give a certain weight to each transition step. Thus d_a^1 gives equal importance to the differences at each transition step, while d_a^0 is the measure that only looks to the immediate transitions. Notice also that for $a \in \mathbb{A}$ the values of d_a^c are of type $k_0 + k_1 \cdot c + k_2 \cdot c^2 + ...$ where k_i are multiples of $\iota(a)$. This is not particularly significant, however, as our main issue is not the absolute value of the metric, but properties like the significance of zero distance or the relative distance of processes.

Consider now the processes $\tau_3.0$, $\tau_2.\tau_1.0$, $\tau_1.\tau_2.0$ and $\tau_1.\tau_1.\tau_1.0$ represented bellow. Their relative distances are:
$d^c(\tau_3.0, \tau_2.\tau_1.0) = |3-2| + c \cdot |1-0| = 1 + c$,
$d^c(\tau_3.0, \tau_1.\tau_2.0) = |3-1| + c \cdot |2-0| = 2 + 2c$,
$d^c(\tau_1.\tau_2.0, \tau_2.\tau_1.0) = |1-2| + c \cdot |2-1| = 1 + c$,
$d^c(\tau_3.0, \tau_1.\tau_1.\tau_1.0) = |3-1| + c \cdot |1-0| + c^2 \cdot |1-0| = 2 + c + c^2$,
$d^c(\tau_2.\tau_1.0, \tau_1.\tau_1.\tau_1.0) = |2-1| + c \cdot |1-1| + c^2 \cdot |1-0| = 1 + c^2$,
$d^c(\tau_1.\tau_2.0, \tau_1.\tau_1.\tau_1.0) = |1-1| + c \cdot |2-1| + c^2 \cdot |1-0| = c + c^2$,

$$\begin{tikzcd}
\tau_3.0 \arrow[d, "{\tau,3}"'] & \tau_2.\tau_1.0 \arrow[d, "{\tau,2}"'] & \tau_1.\tau_2.0 \arrow[d, "{\tau,1}"'] & \tau_1.\tau_1.\tau_1 0 \arrow[d, "{\tau,1}"'] \\
0^\sim & \tau_1.0^\sim \arrow[d, "{\tau,1}"'] & \tau_2.0^\sim \arrow[d, "{\tau,2}"'] & \tau_1.\tau_1.0^\sim \arrow[d, "{\tau,1}"'] \\
 & 0^\sim & 0^\sim & \tau_1.0^\sim \arrow[d, "{\tau,1}"'] \\
 & & & 0^\sim
\end{tikzcd}$$

The next lemma states that, indeed, our functions are pseudometrics.

Lemma 7.1: For any $c \in [0,1]$ and any $\alpha \in \mathbb{A}^+$, d_α^c is a pseudometric on $\mathbb{P}$.

The next theorem states that the distance between bisimilar processes is always zero. It also says that if for a fixed $c \neq 0$ the distances d_α^c between two given processes are zero for all $\alpha \in \mathbb{A}^+$, then the processes are bisimilar.

Theorem 7.1: Let $P, Q \in \mathbb{P}$.
(i) If $P \sim Q$, then for any $c \in [0,1]$ and any $\alpha \in \mathbb{A}^+$, $d_\alpha^c(P, Q) = 0$.
(ii) If there exists $c \in (0,1]$, such that for any $\alpha \in \mathbb{A}^+$, $d_\alpha^c(P, Q) = 0$, then for any $c' \in [0,1]$ and any $\alpha \in \mathbb{A}^+$ $d_\alpha^{c'}(P, Q) = 0$ and, moreover, $P \sim Q$.

Notice that the elements of $\mathbb{D}_\alpha$ measure only α-transitions and for this reason their utility is limited. Our main intention is to introduce a metric on processes that can characterize the bisimulation. For achieving this goal, in what follows we will introduce a family of metrics which consider all the transitions. The intuition is that the "general" distance d^c between two processes is the supremum of the distances d^c_α for all $\alpha \in \mathbb{A}^+$.

Definition 7.2: Consider the family $\mathbb{D}$ of functions $d^c : \mathbb{P}\times\mathbb{P} \to \mathbb{R}^+$ with $c \in [0,1]$, defined for arbitrary $P', P'' \in \mathbb{P}$ by

$$d^c(P',P'') =^{sup}_{\alpha\in\mathbb{A}^+} d^c_\alpha(P',P'').$$

Consider the processes $P = a.a.0 + \tau_r.\tau_r.0$ and $Q = a.(a.0 + a.0) + (\tau_r.0|\tau_r.0)$ represented bellow. For calculating $d^c(P,Q)$, we first observe that $d^c_a(P,Q) = |\iota(a) - \iota(a)| + c \cdot |2\iota(a) - \iota(a)| = c \cdot \iota(a)$, $d^c_\tau(P,Q) = |2r - r| + c \cdot |r - r| = r$ and for any $\alpha \notin \{a, \tau\}$, $d^c_\alpha(P,Q) = 0$.

$$\begin{tikzcd}
 & P \arrow[dl, "{a,\iota(a)}"'] \arrow[dr, "{\tau,r}"] & & & Q \arrow[dl, "{a,\iota(a)}"'] \arrow[dr, "{\tau,2r}"] & \\
a.0^\sim \arrow[d, "{a,\iota(a)}"'] & & \tau_r.0^\sim \arrow[d, "{\tau,r}"'] & a.0+a.0^\sim \arrow[d, "{a,2\iota(a)}"'] & & \tau_r.0^\sim \arrow[d, "{\tau,r}"'] \\
0^\sim & & 0^\sim & 0^\sim & & 0^\sim
\end{tikzcd}$$

Consequently, $d^c(P,Q) = max\{c \cdot \iota(a), r\}$.

Lemma 7.2: For any $c \in [0,1]$, d^c is a pseudometric on $\mathbb{P}$.

The next theorem states that, indeed, the pseudometrics d^c generalise the bisimulation of processes: lifted on the level of bisimulation classes are metrics and consequently organize the space $\mathbb{P}^\sim$ as a metric space.

Theorem 7.2: Let $P, Q \in \mathbb{P}$.
(i) If $P \sim Q$, then for any $c \in [0,1]$, $d^c(P,Q) = 0$.
(ii) If for some $c \in (0,1]$, $d^c(P,Q) = 0$, then for any $c' \in [0,1]$ $d^{c'}(P,Q) = 0$ and $P \sim Q$.

For concluding this section, we notice that the metrics are influenced by the algebraic structure of the processes. The next lemma reveals such a relation for the case of prefixing. However, we believe that more complex relations can be identified and we intend to return to this problem in future works. The possibility of computing the distance between two processes from the relative distances of their sub-processes is an idea that can find interesting applications especially in the case of large systems where it is more convenient to focus on subsystems.

Lemma 7.3: For arbitrary $P, Q \in \mathbb{P}$ and $\epsilon \in \mathbb{A}^*$, if $d^c(P,Q) = r$, then $d^c(\epsilon.P, \epsilon.Q) = max\{2 \cdot \iota(\epsilon), c \cdot r\}$.

VIII. Related work

There has been considerable of work on probabilistic and stochastic process algebras. *Probabilistic process algebras* solve the non-determinism by labeling the transitions with probabilities [33], [25], [3]. In extension, *stochastic process algebras* such as TIPP [16], PEPA [18], [19], EMPA [4], stochastic π-calculus [29] and interactive Markov chain algebra [21], [7] prefix the transitions with probabilistic distributions. In all these cases the transitions are defined pointwise, meaning that the semantics can be described in terms of continuous-time Markov chains and the probabilistic distributions are on the discrete space of processes. For correctly describing the stochastic behaviours with SOS rules of type $P \overset{label}{\longrightarrow} Q$, these calculi involve complex mathematical machineries for labeling and counting. By contrast, our stochastic process algebra is based on the measurable space of processes generated by the structural congruence classes. Our rules are of type $P \longrightarrow \mu$ where μ is a class of distributions on the measurable space of processes. This allows us to proposes an elegant SOS, similar to those of non-deterministic PAs, that maps process-results into distribution-results.

The idea of defining probabilistic transitions by functions that associate to each state of a system a probability distribution over the state space has been considered previously and advocated in the context of probabilistic automata [24], [31]. More recently, the *transition-systems-as-coalgebras* paradigm [10], [30] exploits the same idea, providing a general and uniform mathematical characterisation of transition systems.

The underlying Markovian structures used by our process algebra are more general than continuous-time Markov chains [21] due to the structure imposed by the equational theory. We need a notion of a Markov process defined for a general measurable space and continuous time. For this reason, we use a version of Markovian process similar to the one defined in [14]. Markov processes for arbitrary analytic spaces have been studied by Panangaden et al. in a series of papers [5], [8], [13], [12], [28] where also a notion of stochastic bisimulation, that extends the probabilistic bisimulation of [25], is defined and studied. A similar probabilistic model – Harsanyi type space – has been studied in the context of beliefs systems [17], [20], [27]. Our definition of MP combines these two concepts, relying on results from [15], [28]. On the lines of [28], [12] we define the bisimulation of MPs.

The theory of GSOS [32] has been extended for the case of stochastic systems in [22], [9] where general congruence formats for stochastic GSOS (SGSOS) are studied. The SGSOS framework, as well as GSOS, focuses on the *monads freely generated* by the algebraic signature of a process calculus. Our case is different: we have an *equational monad* because the structural congruence provides extra structure for the class of processes and thus we get a different type of SOS. In our format, for instance, the algebraic signature of processes is different from the algebraic signature of behaviors. Using a σ-algebra which is not the powerset makes our approach different, while considering the measurable sets closed to some congruence relation makes it more appropriate for modeling and for extensions to other *equational theories*.

Metrics for measuring the similarity of probabilistic systems in terms of behaviours have been proposed in [11], [28] following an idea expounded in [23]. These metrics are similar to Kantorovich metric on distributions with the differences that instead of Lipschitz functions, a set of functional expressions that generalise the formulas of Hennessy-Milner logic are used. These metrics are very general being designed for continuous-space Markovian processes. Our metrics are similar to Desharnais-Panangaden metrics, for instance in the way they explore the transition systems, but they are simpler being particularly designed for our process algebra. We do not consider any functional expressions for calculating the distance, but we propose a direct approach. An other important difference is that our metrics measure the differences between processes in terms of rates and not in terms of probabilities.

IX. Concluding remarks

In this paper we develop a stochastic extension of CCS. We propose a structural operational semantics based on measure theory and particularly suited to a domain where a measure of similarity of behaviours is important. For organizing the set of processes as a measurable space, we have chosen the σ-algebra generated by the structural congruence classes of processes and we base the theory on top of it. This choice is motivated by practical modelling reasons: the calculus is meant to be used for applications in computational systems biology. In this context, the structural congruence and the distributions over the space of congruence classes play a key role. The congruence classes represent chemical "soups" and the various syntactic representations of the same soup need to be identified. In fact, structural congruence was inspired by a chemical analogy [2].

The stochastic behaviour is defined using a general concept of Markov process that encapsulates most of the Markovian models, including continuous ones, as well as other models of probabilistic systems, e.g., Harsanyi type spaces. This concept is based on unspecified analytic (hence, measurable) spaces and generalizes rate transition systems [22], [9]. Consequently, we obtain a general definition of stochastic bisimulation similar to the one used in [14].

We also define quantitative extensions of stochastic bisimulation in the form of two classes of metrics that measure the distance between processes in terms of similar behaviours: two processes are at distance zero iff they are bisimilar; two processes are close if their behaviours are similar.

The novelty of this work consists in the fact that the measurable space of processes is axiomatized by structural congruence and the operational semantics reflects the interrelation between this space and the space of distributions on it. Our technology is appropriate for practical modelling purposes where various congruences can be relevant. It will help design (more complex) stochastic process algebras in a uniform way, possibly involving different equational axiomatizations, while avoiding the heavy techniques for counting of reductions. The organisation of the space of processes as a metric space is also a novelty. It can be extended to other calculi and used in applications, for example, to appreciate the quality of approximations of models or to characterise quantitatively the concept of robustness. For future work we intend to extend this calculus to include other algebraical operations, such as recursion or the new name quantification, and to define a general SOS format for these calculi. Another research direction that we intend to follow is the logical characterisation of bisimulation and of the metrics.

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APPENDIX

In this appendix we have collected some of the proofs of the main results presented in the paper.

Proof of Theorem 4.1: It is sufficient to show that for each $P \in \mathbb{P}$ and each $\alpha \in \mathbb{A}^+$, $\theta(\alpha)(P) : \Pi \to \mathbb{R}^+$ is a measure on the measurable space $(\mathbb{P}, \Pi)$. The proof follows the inductive steps of the construction in Definition 4.3. We only treat the non trivial cases.

Case 3: If $P = Q + R$, then $\theta(\alpha)(Q + R)$ is a measure.

First notice that for any $\alpha \in \mathbb{A}$, $\theta(\alpha)(Q + R)(\emptyset) = \theta(\alpha)(Q)(\emptyset) + \theta(\alpha)(R)(\emptyset) = 0$.

Consider now an arbitrary sequence of pairwise disjoint sets $(\mathcal{R}_i)_{i\in I} \in \Pi$. Then, for arbitrary $\alpha \in \mathbb{A}$, $\theta(\alpha)(Q + R)(\cup_{i\in I}\mathcal{R}_i) = \theta(\alpha)(Q)(\cup_{i\in I}\mathcal{R}_i) + \theta(\alpha)(R)(\cup_{i\in I}\mathcal{R}_i)$. The inductive hypothesis guarantees that $\theta(\alpha)(Q)(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I}\theta(\alpha)(Q)(\mathcal{R}_i)$ and $\theta(\alpha)(R)(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I}\theta(\alpha)(R)(\mathcal{R}_i)$. Consequently, $\theta(\alpha)(Q + R)(\cup_{i\in I}\mathcal{R}_i) = \sum_{i\in I}\theta(\alpha)(Q)(\mathcal{R}_i) + \sum_{i\in I}\theta(\alpha)(R)(\mathcal{R}_i) = \sum_{i\in I}(\theta(\alpha)(Q)(\mathcal{R}_i)+\theta(\alpha)(R)(\mathcal{R}_i)) = \sum_{i\in I}\theta(\alpha)(Q + R)(\mathcal{R}_i)$.

Case 4: $P \equiv Q|R$.

Let $a \in \mathbb{A}$. $\theta(a)(Q|R)(\emptyset) = \theta(a)(R)(\emptyset_Q)+\theta(a)(Q)(\emptyset_R) = 0$, because $\emptyset_Q = \emptyset_R = \emptyset$ and, from the inductive hypothesis, $\theta(a)(R)$ and $\theta(a)(Q)$ are measures. Moreover,

$$\theta(\tau)(Q|R)(\emptyset) = \theta(\tau)(R)(\emptyset_Q) + \theta(\tau)(Q)(\emptyset_R)+$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\emptyset}^{a\in\mathbb{A}} \frac{\theta(a)(Q)(\mathcal{P}_1)\cdot\theta(\overline{a})(R)(\mathcal{P}_2)}{2\cdot\iota(a)}.$$

But $\emptyset_Q = \emptyset_R = \emptyset$ and $\mathcal{P}_1|\mathcal{P}_2 \subseteq \emptyset$ implies $\mathcal{P}_1 = \mathcal{P}_2 = \emptyset$. The inductive hypothesis guarantees that $\theta(\tau)(R)(\emptyset) = \theta(\tau)(Q)(\emptyset) = \theta(\overline{a})(R)(\emptyset) = \theta(a)(Q)(\emptyset) = 0$. Hence, $\theta(Q|R)(\tau)(\emptyset) = 0$.

Consider now an arbitrary sequence of pairwise disjoint sets $(\mathcal{R}^i)_{i\in I} \in \Pi$ and let $\mathcal{P} = \cup_{i\in I}\mathcal{R}^i$. Then,

$$\theta(a)(Q|R)(\mathcal{P}) = \theta(a)(R)(\mathcal{P}_Q) + \theta(a)(Q)(\mathcal{P}_R).$$

Observe that $\mathcal{R}^i_Q$ and $\mathcal{R}^i_R$ are pairwise disjoint, because the sets $\mathcal{R}^i$ are pairwise disjoint. Consequently, using the inductive hypothesis, we obtain

$$\theta(a)(Q|R)(\mathcal{P}) = \sum_{i\in I}\theta(a)(R)(\mathcal{R}^i_Q) + \sum_{i\in I}\theta(a)(Q)(\mathcal{R}^i_R) =$$

$$\sum_{i\in I}[\theta(a)(R)(\mathcal{R}^i_Q) + \theta(a)(Q)(\mathcal{R}^i_R)] = \sum_{i\in I}\theta(a)(Q|R)(\mathcal{R}^i).$$

$$\theta(\tau)(Q|R)(\mathcal{P}) = \theta(\tau)(R)(\mathcal{P}_Q) + \theta(\tau)(Q)(\mathcal{P}_R)+$$

$$+\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{P}}^{a\in\mathbb{A}} \frac{\theta(a)(Q)(\mathcal{P}_1)\cdot\theta(\overline{a})(R)(\mathcal{P}_2)}{2\cdot\iota(a)}.$$

As before,

$$\theta(\tau)(Q|R)(\mathcal{P}) = \sum_{i\in I}\theta(\tau)(R)(\mathcal{R}^i_Q) + \sum_{i\in I}\theta(\tau)(Q)(\mathcal{R}^i_R)+$$

$$\sum_{i\in I}\sum_{\mathcal{P}_1|\mathcal{P}_2\in\mathcal{R}_i}^{a\in\mathbb{A}} \frac{\theta(a)(Q)(\mathcal{P}_1)\cdot\theta(\overline{a})(R)(\mathcal{P}_2)}{2\cdot\iota(a)} =$$

$$= \sum_{i\in I}\theta(\tau)(Q|R)(\mathcal{R}_i).$$

■

Proof of Lemma 5.1: We only prove 2(b) and 2(c), the other cases being trivial.

2(b). Let $\mu = \mu' \;_{P}\!\otimes_Q\, \mu''$ and arbitrary $a \in \mathbb{A}$, $\mathcal{R} \in \Pi$.

$$((\mu' \;_{P}\!\otimes_Q\, \mu'') \;_{P|Q}\!\otimes_R\, \mu''')(a)(\mathcal{R}) = (\mu \;_{P|Q}\!\otimes_R\, \mu''')(a)(\mathcal{R}) =$$

$$\mu(a)(\mathcal{R}_R) + \mu'''(a)(\mathcal{R}_{P|Q})$$

But $\mu(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}) \;_{P}\!\otimes_Q\, \mu''(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}_Q) + \mu''(a)(\mathcal{R}_P)$.

$$((\mu' \;_{P}\!\otimes_Q\, \mu'') \;_{P|Q}\!\otimes_R\, \mu''')(a)(\mathcal{R}) =$$

$$= (\mu'(a)((\mathcal{R}_R)_Q) + \mu''(a)((\mathcal{R}_R)_P)) + \mu'''(a)(\mathcal{R}_{P|Q}).$$

Observe that for arbitrary $P, Q \in \mathbb{P}$ and arbitrary $\mathcal{R} \in \Pi$, $(\mathcal{R}_P)_Q = \mathcal{R}_{P|Q}$. Using this, we obtain

$$((\mu' \;_{P}\!\otimes_Q\, \mu'') \;_{P|Q}\!\otimes_R\, \mu''')(a)(\mathcal{R}) =$$

$$= \mu'(a)(\mathcal{R}_{Q|R}) + \mu''(a)(\mathcal{R}_{P|R}) + \mu'''(a)(\mathcal{R}_{P|Q}).$$

In the same way we can prove that

$$\mu' \;_{P}\!\otimes_{Q|R}\, (\mu'' \;_{Q}\!\otimes_R\, \mu''')(a)(\mathcal{R}) =$$

$$= \mu'(a)(\mathcal{R}_{Q|R}) + \mu''(a)(\mathcal{R}_{P|R}) + \mu'''(a)(\mathcal{R}_{P|Q}).$$

Now we prove that

$$((\mu' \;_{P}\!\otimes_Q \mu'') \;_{P|Q}\!\otimes_R \mu''')(\tau)(\mathcal{R}) = (\mu' \;_{P}\!\otimes_{Q|R}(\mu'' \;_{Q}\!\otimes_R \mu'''))(\tau)(\mathcal{R}).$$

As before, we have

$$((\mu' \;_{P}\!\otimes_Q\, \mu'') \;_{P|Q}\!\otimes_R\, \mu''')(\tau)(\mathcal{R}) = (\mu \;_{P|Q}\!\otimes_R\, \mu''')(\tau)(\mathcal{R}) =$$

$$= \mu(\tau)(\mathcal{R}_R)+\mu'''(\tau)(\mathcal{R}_{P|R})+\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2\cdot\iota(a)} =$$

$$= \mu'(\tau)(\mathcal{R}_{R|Q}) + \mu''(\tau)(\mathcal{R}_{R|P}) + \mu'''(\tau)(\mathcal{R}_{P|R})+$$

$$\sum_{\mathcal{Q}_1|\mathcal{Q}_2\subseteq\mathcal{R}_R}^{b\in\mathbb{A}} \frac{\mu'(b)(\mathcal{Q}_1)\cdot\mu''(\overline{b})(\mathcal{Q}_2)}{2\cdot\iota(b)} + \sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{2\cdot\iota(a)}.$$

But

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} =$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{[\mu'(a)((\mathcal{P}_1)_Q) + \mu''(a)((\mathcal{P}_1)_P)]\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} =$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu'(a)((\mathcal{P}_1)_Q)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2}+$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu''(a)((\mathcal{P}_1)_P)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2}.$$

Observe that, due to the way the sum is defined (and because pairing is an involution) we have that

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu'(a)((\mathcal{P}_1)_Q)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} =$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_Q}^{a\in\mathbb{A}} \frac{\mu'(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2}$$

and

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu''(a)((\mathcal{P}_1)_P)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} =$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_P}^{a\in\mathbb{A}} \frac{\mu''(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2}.$$

Consequently,

$$((\mu' \ {}_P\!\otimes_Q \mu'') \ {}_{P|Q}\!\otimes_R \mu''')(\tau_{(r)})(\mathcal{R}) =$$

$$= \mu'(\tau)(\mathcal{R}_{R|Q}) + \mu''(\tau)(\mathcal{R}_{R|P}) + \mu'''(\tau)(\mathcal{R}_{P|R}) +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_R}^{a\in\mathbb{A}} \frac{\mu'(a)(\mathcal{P}_1)\cdot\mu''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_Q}^{a\in\mathbb{A}} \frac{\mu'(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_P}^{a\in\mathbb{A}} \frac{\mu''(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2}.$$

Similarly can be proved that

$$(\mu' \ {}_P\!\otimes_{Q|R} (\mu'' \ {}_Q\!\otimes_R \mu'''))(\tau)(\mathcal{R}) =$$

$$= \mu'(\tau)(\mathcal{R}_{R|Q}) + \mu''(\tau)(\mathcal{R}_{R|P}) + \mu'''(\tau)(\mathcal{R}_{P|R}) +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_R}^{a\in\mathbb{A}} \frac{\mu'(a)(\mathcal{P}_1)\cdot\mu''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_Q}^{a\in\mathbb{A}} \frac{\mu'(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2} +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}_P}^{a\in\mathbb{A}} \frac{\mu''(a)(\mathcal{P}_1)\cdot\mu'''(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2}.$$

3(c). We prove now that $\mu' \ {}_P\!\otimes_0 \overline{\omega} = \mu'$. Consider arbitrary $a \in \mathbb{A}$ and $\mathcal{R} \in \Pi$.

$$(\mu' \ {}_P\!\otimes_0 \overline{\omega})(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}_0) + \overline{\omega}(a)(R_P).$$

But $\mathcal{R}_0 = \mathcal{R}$ and $\overline{\omega}(a)(R_P) = 0$. Consequently,

$$(\mu' \ {}_P\!\otimes_0 \overline{\omega})(a)(\mathcal{R}) = \mu'(a)(\mathcal{R}).$$

We also have

$$(\mu' \ {}_P\!\otimes_0 \overline{\omega})(\tau)(\mathcal{R}) = \mu'(\tau)(\mathcal{R}_0) + \overline{\omega}(\tau)(R_P) +$$

$$\sum_{\mathcal{P}_1|\mathcal{P}_2\subseteq\mathcal{R}}^{a\in\mathbb{A}} \frac{\mu'(a)(\mathcal{P}_1)\cdot\overline{\omega}(\overline{a})(\mathcal{P}_2)}{\iota(a)\cdot 2}.$$

But $\overline{\omega}(\overline{a})(\mathcal{P}_1) = \overline{\omega}(\tau)(\mathcal{R}_P) = 0$ and $\mathcal{R}_0 = \mathcal{R}$, where from we obtain

$$(\mu' \ {}_P\!\otimes_0 \overline{\omega})(\tau)(\mathcal{R}) = \mu'(\tau)(\mathcal{R}).$$

■

Proof of Lemma 5.3: The proof is done by induction on the structures of P and Q following the axioms of the structural congruence.

The case $P = R'|S$, $Q = R''|S$ with $R' \equiv R''$. Suppose that $S \to \mu'$ and $R' \to \mu''$ (from the inductive hypothesis, $R'' \to \mu''$). Then, $\mu = \mu'' \ {}_{R'}\!\otimes_S \mu'$. Using 3(a) of Lemma 5.1, we obtain that $\mu = \mu'' \ {}_{R''}\!\otimes_S \mu'$ and, by (Par), $Q \to \mu$.

The case $P = R' + S$, $Q = R'' + S$ with $R' \equiv R''$. Suppose that $S \to \mu'$ and $R' \to \mu''$ (from the inductive hypothesis, $R'' \to \mu''$). Then, by (Sum), $Q \to \mu'' \oplus \mu'$. But $\mu = \mu'' \oplus \mu'$.

The case $P = \alpha.R$, $Q = \alpha.S$ with $R \equiv S$. We have $\mu = \alpha_R$ and $S \to \alpha_S$. As $R \equiv S$, we obtain that $\mu = \alpha_S$, i.e., $Q \to \mu$.

The case $P = R|S$, $Q = S|R$. Suppose that $R \to \mu'$ and $S \to \mu''$. Then $Q \to \mu'' \ {}_S\!\otimes_R \mu'$ and $\mu = \mu' \ {}_R\!\otimes_S \mu''$. But we proved in Lemma 5.1 that $\mu'' \ {}_S\!\otimes_R \mu' = \mu' \ {}_R\!\otimes_S \mu''$.

The case $P = (R|S)|T$, $Q = R|(S|T)$. Suppose that $R \to \mu'$, $S \to \mu''$ and $T \to \mu'''$. Then $Q \to \mu' \ {}_R\!\otimes_{S|T} (\mu'' \ {}_S\!\otimes_T \mu''')$ and $\mu = (\mu' \ {}_R\!\otimes_S \mu'') \ {}_{R|S}\!\otimes_T \mu'''$. But we proved in Lemma 5.1 that $\mu' \ {}_R\!\otimes_{S|T} (\mu'' \ {}_S\!\otimes_T \mu''') = (\mu' \ {}_R\!\otimes_S \mu'') \ {}_{R|S}\!\otimes_T \mu'''$.

The case $Q = P|0$. $Q \to \mu \ {}_P\!\otimes_0 \overline{\omega}$. But, from Lemma 5.1, $\mu \ {}_P\!\otimes_0 \overline{\omega} = \mu$.

The cases [$P = R+S$ and $Q = S+R$], [$P = (R+S)+T$ and $Q = R+(S+T)$] and [$Q = P+0$]. These are consequences of the fact that $(\Delta(\mathbb{P})^{\mathbb{A}^+}, \oplus, \overline{\omega})$ is a commutative monoid (Lemma 5.1). ■

Proof of Theorem 6.1: Before proceeding with the proof let's notice that if we have two equivalence relations $\mathcal{R}_1, \mathcal{R}_2$ on a set M, there exists an equivalence relation $\mathcal{R}$ on M such that $R_1 \cup \mathcal{R}_2 \subseteq \mathcal{R}$. Moreover, each R-equivalence class can be seen as the reunion of $\mathcal{R}_1$-equivalence classes as well as the reunion of $\mathcal{R}_2$-equivalence classes. The same result is true if we start from a denumerable set of equivalence relations.

We prove now that $\sim$ is an equivalence relation. Reflexivity and symmetry are trivial. We prove the transitivity.

Suppose that $P \sim Q$ and $Q \sim R$, $P \to \mu$, $Q \to \mu'$ and $R \to \mu''$. Then, there exist two stochastic bisimulation relations $\mathcal{R}_1, \mathcal{R}_2$ such that $(P, Q) \in \mathcal{R}_1$ and $(Q, R) \in \mathcal{R}_2$. Let $\mathcal{R}$ be the smallest equivalence relation such that $\mathcal{R}_1 \cup \mathcal{R}_2 \subseteq \mathcal{R}$.

Consider arbitrary $\alpha \in \mathbb{A}^+$ and $C \in \Pi(\mathcal{R})$. Observe that, by definition, $\Pi(\mathcal{R}) = \Pi \cap \mathbb{P}^{\mathcal{R}}$, where we denoted by $\mathbb{P}^{\mathcal{R}}$ the set of $\mathcal{R}$-equivalence classes. Hence, $C \in \mathbb{P}^{\mathcal{R}}$ and because $C \in \Pi$ and Π is denumerable, we obtain that there exist $(C_1^i)_{i\in I} \subseteq \mathbb{P}^{\mathcal{R}_1}$ and $(C_2^j)_{j\in J} \subseteq \mathbb{P}^{\mathcal{R}_2}$ at most denumerable sets of $\mathcal{R}_1$ and respectively $\mathcal{R}_2$-equivalence classes, such that

$$C = \bigcup_{i\in I} C_1^i = \bigcup_{j\in J} C_2^j.$$

We also assume that the elements of $(C_1^i)_{i\in I}$ are pairwise distinct hence, (because they are equivalence classes) are pairwise disjoint. The same about $(C_2^j)_{j\in J}$.

Because $(P,Q) \in \mathcal{R}_1$, we have that for each $C_i \in \Pi(\mathcal{R}_1) = \Pi \cap \mathbb{P}^{\mathcal{R}_1}$ and each $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C_i) = \mu'(\alpha)(C_i)$.

Because $(Q,R) \in \mathcal{R}_2$, we have that for each $C_j \in \Pi(\mathcal{R}_2) = \Pi \cap \mathbb{P}^{\mathcal{R}_2}$ and each $\alpha \in \mathbb{A}^+$, $\mu'(\alpha)(C_j) = \mu''(\alpha)(C_j)$.

We show that for each $C \in \Pi(\mathcal{R})$ and each $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C) = \mu''(\alpha)(C)$. Because $\mu(\alpha), \mu'(\alpha)$ and $\mu''(\alpha)$ are measures, we obtain

$$\mu(\alpha)(C) = \sum_{i\in I} \mu(\alpha)(C_i) = \sum_{i\in I} \mu'(\alpha)(C_i) = \mu'(\alpha)(C).$$

Similarly,

$$\mu'(\alpha)(C) = \sum_{j\in J} \mu'(\alpha)(C_j) = \sum_{j\in J} \mu''(\alpha)(C_j) = \mu''(\alpha)(C).$$

Hence, $\mu(\alpha)(C) = \mu''(\alpha)(C)$ proving that $\mathcal{R}$ is a stochastic bisimulation and concluding the transitivity proof.

For showing that $P \sim Q$ iff for any $C \in \Pi(\sim)$ and any $\alpha \in \mathbb{A}$, $\mu(\alpha)(C) = \mu'(\alpha)(C)$, we proceed as before, observing that $P \sim Q$ implies the existence of a bisimulation relation $\mathcal{R}$ such that $(P,Q) \in \mathcal{R}$. We can show that each $C \in \Pi(\sim)$ can be represented as a reunion of (at most denumerable) pairwise disjoint measurable $\mathcal{R}$-equivalence classes and, using the fact that $\mu(\alpha), \mu'(\alpha)$ are measures we can show that $\mu(\alpha)(C) = \mu'(\alpha)(C)$. ∎

Proof of Theorem 6.2: Suppose that $P \to \mu$. $P \equiv Q$ implies (Lemma 5.3) that $Q \to \mu$. As for any $\sim$-equivalence class C and any $\alpha \in \mathbb{A}^+$, $\mu(\alpha)(C) = \mu(\alpha)(C)$, we obtain $P \sim Q$. ∎

Proof of Theorem 6.3:

For the proof of Theorem 6.3 we need the next lemma.

Lemma 9.1: For arbitrary $P, Q, R \in \mathbb{P}$, if $P \sim Q$, then $P|R \sim Q|R$.

Proof: We prove this lemma inductively on the structure of the processes involved. For doing this, we first define the complexity of a process as the number of algebraic operators appearing in its syntax.

Let $cx : \mathbb{P} \to \mathbb{N}$ by
$cx(0) = 0$, $cx(\alpha.P) = cx(P) + 1$ and $cx(P|Q) = cx(P + Q) = cx(P) + cx(Q)$.

Observe that the complexity of a process is strictly related to the behavior of the process. Indeed, if for some $r \neq 0$, $P \xrightarrow{\alpha,r} Q$, then $cx(P) > cx(Q)$.

For $(x_1, x_2), (y_1, y_2) \in \mathbb{N}^2$ we write $(x_1, x_2) < (y_1, y_2)$ iff for each $i = 1, 2$, $x_i \le y_i$ and for some $j = 1, 2$, $x_j < y_j$. With this notation, we will prove the lemma inductively on $(max(cx(P), cx(Q)), cx(R))$.

The base case is trivial, so we prove, in what follows, the inductive step.

Suppose that for any $P', Q', R' \in \mathbb{P}$ with

$$(max(cx(P'), cx(Q')), cx(R')) < (max(cx(P), cx(Q)), cx(R))$$

we have that if $P' \sim Q'$, then $P'|R' \sim Q'|R'$. And we show that if $P \sim Q$, then $P|R \sim Q|R$.

Suppose that $P \to \mu$, $Q \to \eta$ and $R \to \rho$. Then, $P|R \to \mu\ {}_P{\otimes}_R\ \rho$ and $Q|R \to \eta\ {}_Q{\otimes}_R\ \rho$. For showing $P|R \sim Q|R$, it is sufficient to show that for arbitrary $\alpha \in \mathbb{A}^+$ and $C \in \mathbb{P}^{\sim}$,

$$(\mu\ {}_P{\otimes}_R\ \rho)(\alpha)(C) = (\eta\ {}_Q{\otimes}_R\ \rho)(\alpha)(C).$$

The case $\alpha = a \in \mathbb{A}$.

Due to Lemma 5.4 we can assume that:

- there exists a finite set of processes $\mathcal{P} = \{P_1^1, ..., P_1^{n_1}, ..., P_k^1, ..., P_k^{n_k}\}$, pairwise non structural congruent, such that $P \xrightarrow{a,0} \mathbb{P} \setminus \mathcal{P}$ and $P \xrightarrow{a,p_i^j} P_i^j$ for some $p_i^j \neq 0$; in addition, for each $i = 1..k$ and each $j, j' \in \{1, ..n_i\}$, $P_i^j \sim P_i^{j'}$ and for $i \neq i'$, $x = 1..n_i$, $x' = 1..n_{i'}$, $P_i^x \not\sim P_{i'}^{x'}$; let $p_i = \sum_{j=1..n_i} p_i^j$;
- there exists a finite set of processes $\mathcal{Q} = \{Q_1^1, ..., Q_1^{m_1}, ..., Q_l^1, ..., Q_l^{m_l}\}$, pairwise non structural congruent, such that $Q \xrightarrow{a,0} \mathbb{P} \setminus \mathcal{Q}$ and $Q \xrightarrow{a,q_i^j} Q_i^j$ for some $q_i^j \neq 0$; in addition, for each $i = 1..l$ and each $j, j' \in \{1, ..m_i\}$, $Q_i^j \sim Q_i^{j'}$ and for $i \neq i'$, $x = 1..m_i$, $x' = 1..m_{i'}$, $Q_i^x \not\sim Q_{i'}^{x'}$; let $q_i = \sum_{j=1..m_i} q_i^j$;
- there exists a finite set of processes $\mathcal{R} = \{R_1^1, ..., R_1^{u_1}, ..., R_v^1, ..., R_v^{u_v}\}$, pairwise non structural congruent, such that $R \xrightarrow{a,0} \mathbb{P} \setminus \mathcal{R}$ and $R \xrightarrow{a,r_i^j} R_i^j$ for some $r_i^j \neq 0$; in addition, for each $i = 1..v$ and each $j, j' \in \{1, ..u_i\}$, $R_i^j \sim R_i^{j'}$ and for $i \neq i'$, $x = 1..u_i$, $x' = 1..u_{i'}$, $R_i^x \not\sim r_{i'}^{x'}$; let $r_i = \sum_{j=1..u_i} r_i^j$;

Observe that $P \sim Q$ implies $k = l$, we can suppose that $P_i^j \sim Q_i^{j'}$ and for each $i = 1..k$, $p_i = q_i$.

For arbitrary $C \in \mathbb{P}^{\sim}$,

$$(\mu\ {}_P{\otimes}_R\ \rho)(a)(C) = \mu(a)(C_R) + \rho(a)(C_P) = \sum_{(P_1|R)^{\equiv} \subseteq C} \mu(a)(P_1^{\equiv}) + \sum_{(R_1|P)^{\equiv} \subseteq C} \rho(a)(R_1^{\equiv}),$$

and

$$(\eta\ {}_Q{\otimes}_R\ \rho)(a)(C) = \eta(a)(C_R) + \rho(a)(C_Q) = \sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(a)(Q_1^{\equiv}) + \sum_{(R_1|Q)^{\equiv} \subseteq C} \rho(a)(R_1^{\equiv}).$$

If there exist $i_1, ..i_t$ such that for each $i \in \{i_1, ..i_t\}$ and only for them there exist $j \in \{1..n_i\}$ with $P_i^j|R \in C$, then, from the inductive hypothesis we have that for each $j' = 1..n_i$, $P_i^{j'}|R \in C$. Moreover, if $P'|R \in C$ such that $P|R \xrightarrow{a,s}$

$P'|R$ for $s \neq 0$, then there exist i, j such that $P' \equiv P_i^j$. Consequently,

$$\sum_{(P_1|R)^{\equiv} \subseteq C} \mu(a)(P_1^{\equiv}) = \sum_{s=1..t} p_s.$$

But $P_i^j \sim Q_i^{j'}$ where from, using the inductive hypothesis, $Q_i^{j'}|R \in C$. Further, a similar argument as before gives

$$\sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(a)(Q_1^{\equiv}) = \sum_{s=1..t} p_s.$$

On the other hand, if there exist no i and j such that $P_i^j|R \in C$, from $P \sim Q$ we can prove that there is no i, j such that $Q_i^j|R \in C$, where from we obtain

$$\sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(a)(Q_1^{\equiv}) = \sum_{(P_1|R)^{\equiv} \subseteq C} \mu(a)(P_1^{\equiv}) = 0.$$

Observe now that $P \sim Q$ implies, using the inductive hypothesis, that $P|R_i^j \sim Q|R_i^j$, i.e., $R_i^j|P \in C$ iff $Q|R_i^j \in C$. Moreover, if $P|R_i^j \in C$, $P|R_i^{j'} \in C$ for any $j' = 1, ..u_i$ and for any $i' \neq i$, $P|R_{i'}^{j''} \notin C$. Hence, supposing that $R_i^j|P \in C$, we obtain

$$\sum_{(R'|P)^{\equiv} \subseteq C} \rho(a)(R') = \sum_{(R'|Q)^{\equiv} \subseteq C} \rho(a)(R') = r_i.$$

Else, if for no i, j, $R_i^j|P \in C$ we also have that for no i, j $R_i^j|Q \in C$ implying

$$\sum_{(R'|P)^{\equiv} \subseteq C} \rho(a)(R') = \sum_{(R'|Q)^{\equiv} \subseteq C} \rho(a)(R') = 0.$$

The case $\alpha = \tau$.
Due to Lemma 5.4 we can assume that:

- there exists a finite set of processes $\mathcal{P} = \{P_0^1, ..., P_0^{n_0}\}$, pairwise non structural congruent, such that $P \xrightarrow{\tau, p_0^j} P_0^j$ for some $p_0^j \neq 0$ and $P \xrightarrow{\tau, 0} \mathbb{P} \setminus \mathcal{P}$; in addition, there exists a finite set of actions $a \in \mathbb{A}$ with $P \xrightarrow{a, s} \mathbb{P}$ for some $s \neq 0$ and for each such a there exists a set $\{P_1^1, ..., P_1^{n_1}, ..., P_k^1, ..., P_k^{n_k}\}$ of processes, pairwise non structural congruent, such that $P \xrightarrow{a, p_i^j} P_i^j$ for some $p_i^j \neq 0$; moreover, for each $i = 0..k$ and $j, j' \in \{1, ..n_i\}$, $P_i^j \sim P_j^{j'}$ and for $i \neq i'$, $x = 1..n_i$, $x' = 1..n_{i'}$, $P_i^x \not\sim P_{i'}^{x'}$; let $p_i = \sum_{j=1..n_i} p_i^j$ for each $i = 0, ..k$;
- there exists a finite set of processes $\mathcal{Q} = \{Q_0^1, ..., Q_0^{m_0}\}$, pairwise non structural congruent, such that $Q \xrightarrow{\tau, q_0^j} Q_0^j$ for some $q_0^j \neq 0$ and $Q \xrightarrow{\tau, 0} \mathbb{P} \setminus \mathcal{Q}$; in addition, there exists a finite set of actions $a \in \mathbb{A}$ with $Q \xrightarrow{a, s} \mathbb{P}$ for some $s \neq 0$ and for each such a there exists a set $\{Q_1^1, ..., Q_1^{m_1}, ..., P_l^1, ..., P_l^{m_l}\}$ of processes, pairwise non structural congruent, such that $Q \xrightarrow{a, q_i^j} Q_i^j$ for some $q_i^j \neq 0$; moreover, for each $i = 0..l$ and $j, j' \in \{1, ..m_i\}$, $Q_i^j \sim Q_j^{j'}$ and for $i \neq i'$, $x = 1..m_i$, $x' = 1..m_{i'}$, $Q_i^x \not\sim Q_{i'}^{x'}$; let $q_i = \sum_{j=1..n_i} q_i^j$ for each $i = 0, ..l$;
- there exists a finite set of processes $\mathcal{R} = \{R_0^1, ..., R_0^{u_0}\}$, pairwise non structural congruent, such that $R \xrightarrow{\tau, r_0^j} R_0^j$ for some $r_0^j \neq 0$ and $R \xrightarrow{\tau, 0} \mathbb{P} \setminus \mathcal{R}$; in addition, there exists a finite set of actions $a \in \mathbb{A}$ with $R \xrightarrow{a, s} \mathbb{P}$ for some $s \neq 0$ and for each such a there exists a set $\{R_1^1, ..., R_1^{n_1}, ..., R_k^1, ..., R_k^{n_k}\}$ of processes, pairwise non structural congruent, such that $R \xrightarrow{a, r_i^j} R_i^j$ for some $r_i^j \neq 0$; moreover, for each $i = 0..v$ and $j, j' \in \{1, ..u_i\}$, $R_i^j \sim R_j^{j'}$ and for $i \neq i'$, $x = 1..u_i$, $x' = 1..u_{i'}$, $R_i^x \not\sim R_{i'}^{x'}$; let $r_i = \sum_{j=1..n_i} p_i^j$ for each $i = 0, ..v$;

Observe that $P \sim Q$ implies, for each a having the mentioned properties, that $k = l$; we can suppose, without loosing generality, that $P_i^j \sim Q_i^{j'}$ and for each $i = 0..k$, $p_i = q_i$.

For arbitrary $C \in \mathbb{P}^{\sim}$,

$$(\mu \;{}_P\!\otimes_R\, \rho)(\tau)(C) = \mu(\tau)(C_R) + \rho(\tau)(C_P) + \sum_{(P_1|P_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}} \frac{\mu(a)(P_1^{\equiv}) \cdot \rho(\overline{a})(P_2^{\equiv})}{\iota(a) \cdot 2} =$$

$$\sum_{(P_1|R)^{\equiv} \subseteq C} \mu(\tau)(P_1^{\equiv}) + \sum_{(R_1|P)^{\equiv} \subseteq C} \rho(\tau)(R_1^{\equiv}) + \sum_{(P_1|P_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}} \frac{\mu(a)(P_1^{\equiv}) \cdot \rho(\overline{a})(P_2^{\equiv})}{\iota(a) \cdot 2},$$

and

$$(\eta \;{}_Q\!\otimes_R\, \rho)(a)(C) = \eta(\tau)(C_R) + \rho(\tau)(C_Q) + \sum_{(Q_1|Q_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}} \frac{\eta(a)(Q_1^{\equiv}) \cdot \rho(\overline{a})(Q_2^{\equiv})}{\iota(a) \cdot 2} =$$

$$\sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(\tau)(Q_1^{\equiv}) + \sum_{(R_1|Q)^{\equiv} \subseteq C} \rho(\tau)(R_1^{\equiv}) + \sum_{(Q_1|Q_2)^{\equiv} \subseteq C}^{a \in \mathbb{A}} \frac{\eta(a)(Q_1^{\equiv}) \cdot \rho(\overline{a})(Q_2^{\equiv})}{\iota(a) \cdot 2}.$$

At this level we can demonstrate, using the same strategy as in the case $\alpha = a$, that

- $$\sum_{(P_1|R)^{\equiv} \subseteq C} \mu(\tau)(P_1^{\equiv}) = \sum_{(Q_1|R)^{\equiv} \subseteq C} \eta(\tau)(Q_1 \equiv) = \sum_{i=1..t} p_i,$$
  where $i_1, ..i_t$ are such that for each $i \in \{i_1, ..i_t\}$, there exists some j such that (hence, for all j) $P_i^j|R \in C$ and, from the inductive hypothesis, there exists j' such that (hence, for all j') $Q_i^{j'}|R \in C$;
- because $P|R_i^j \sim Q|R_i^j$,
  $$\sum_{(R_1|P)^{\equiv} \subseteq C} \rho(\tau)(R_1^{\equiv}) = \sum_{(R_1|Q)^{\equiv} \subseteq C} \rho(\tau)(R_1^{\equiv}) = r_i,$$
  where i is (the unique index) such that for some (hence, for all) j, j', $P|R_i^j, Q|R_i^{j'} \in C$.

- for each a as before we also have

$$\sum_{(P_1|P_2)^{\equiv}\subseteq C} \frac{\eta(a)(P_1^{\equiv})\cdot\rho(\overline{a})(P_2^{\equiv})}{\iota(a)\cdot 2} = \sum_{(i,j)\in I} p_i\cdot q_j =$$

$$= \sum_{(P_1|P_2)^{\equiv}\subseteq C} \frac{\eta(a)(P_1^{\equiv})\cdot\rho(\overline{a})(P_2^{\equiv})}{\iota(a)\cdot 2},$$

where I is the set of pairs of indexes (i,j) such that for some x,y (hence, for all), $P_i^x|Q_j^y\in C$.

■

We can proceed now with the proof of Theorem 6.3.

1. If $P\sim P'$, we have that for any $C\in\mathbb{P}^{\sim}$, $P\in C$ iff $P'\in C$. From here, we derive that $[^{\epsilon}_{P}](\alpha)(C)=[^{\epsilon}_{P'}](\alpha)(C)$.

2. If $P\sim P'$ and $Q\sim Q'$, then $P+Q\sim P'+Q'$. Suppose that $P\to\mu$, $P'\to\mu'$, $Q\to\eta$ and $Q'\to\eta'$. Consider an arbitrary $C\in\mathbb{P}^{\sim}$.

For $\alpha\in\mathbb{A}^+$, $(\mu\oplus\eta)(\alpha)(C)=\mu(\alpha)(C)+\eta(\alpha)(C)$. But $P\sim P'$ and $Q\sim Q'$, i.e. for any $C\in\mathbb{P}^{\sim}$, $\mu(\alpha)(C)=\mu'(\alpha)(C)$ and $\eta(\alpha)(C)=\eta'(\alpha)(C)$. Hence, $\mu(\alpha)(C)+\eta(\alpha)(C)=\mu'(\alpha)(C)+\eta'(\alpha)(C)=(\mu'\oplus\eta')(\alpha)(C)$.

3. If $P\sim P'$ and $Q\sim Q'$, then $P|Q\sim P'|Q'$. Using Lemma 9.1, we obtain that $P\sim P'$ implies $P|P'\sim Q|P'$ and $Q\sim Q'$ implies $Q|P'\sim Q'|P'$. Further, the transitivity of $\sim$ proves $P|Q\sim P'|Q'$. ■

Proof of Lemma 7.1: The only non-trivial axiom of pseudometrics that we need to prove is: $d^c_\alpha(P,R)\leq d^c_\alpha(P,Q)+d^c_\alpha(Q,R)$. We prove it by induction on the structures of processes. We can assume, without loosing generality, that there exist the processes $P_1,...,P_n$, $Q_1,...,Q_n$ and $R_1,...,R_n$ such that $\mathcal{D}(P)\subseteq\{P_1,...,P_n\}$, $\mathcal{D}(Q)\subseteq\{Q_1,...,Q_n\}$ and $\mathcal{D}(R)\subseteq\{R_1,...,R_n\}$ and $\sigma',\sigma'\in\mathcal{B}$ such that $d^c_\alpha(P,Q)=\sigma'_\alpha(P,Q)$, $d^c_\alpha(Q,R)=\sigma''_\alpha(Q,R)$ and for each $i=1..n$, $Q_i^{\sim}=\sigma'(P_i^{\sim})$, $R_i^{\sim}=\sigma''(Q_i^{\sim})$. Suppose also that $P\to\mu$, $Q\to\mu'$ and $R\to\mu''$. Then, $\sigma'''=\sigma''\circ\sigma'\in\mathcal{B}$. Consequently, $d^c_\alpha(P,R)\leq\sigma'''_\alpha(P,R)=$

$$\sum_{i=1..n}(|\mu(\alpha)(P_i^{\sim})-\mu''(\alpha)(R_i^{\sim})|+c\cdot d^c_\alpha(P_i,R_i)).$$

From the inductive hypothesis we obtain that for each $i=1..n$, $d^c_\alpha(P_i,R_i)\leq d^c_\alpha(P_i,Q_i)+d^c_\alpha(Q_i,R_i)$. Moreover, $|\mu(\alpha)(P_i)-\mu''(\alpha)(R_i)|\leq|\mu(\alpha)(P_i)-\mu'(\alpha)(Q_i)|+|\mu'(\alpha)(Q_i)-\mu''(\alpha)(R_i)|$. These imply that

$$\sum_{i=1..n}(|\mu(\alpha)(P_i^{\sim})-\mu''(\alpha)(R_i^{\sim})|+c\cdot d^c_\alpha(P_i,R_i))\leq$$

$$\sum_{i=1..n}(|\mu(\alpha)(P_i^{\sim})-\mu'(\alpha)(Q_i^{\sim})|+c\cdot d^c_\alpha(P_i,Q_i))+$$

$$\sum_{i=1..n}(|\mu'(\alpha)(Q_i^{\sim})-\mu''(\alpha)(R_i^{\sim})|+c\cdot d^c_\alpha(Q_i,R_i))=$$

$$\sigma'_\alpha(P,Q)+\sigma''_\alpha(Q,R)=d^c_\alpha(P,Q)+d^c_\alpha(Q,R).$$

Hence, $d^c_\alpha(P,R)\leq d^c_\alpha(P,Q)+d^c_\alpha(Q,R)$. ■

Proof of Lemma 7.2: As for d^c_α the only non-trivial axiom to verify is $d^c(P,R)\leq d^c(P,Q)+d^c(Q,R)$. In Lemma 7.1 we have proved that $d^c_\alpha(P,R)\leq d^c_\alpha(P,Q)+d^c_\alpha(Q,R)$. From here we obtain

$$d^c(P,R)=^{sup}_{\alpha\in\mathbb{A}^+}d^c_\alpha(P,R)\leq^{sup}_{\alpha\in\mathbb{A}^+}(d^c_\alpha(P,Q)+d^c_\alpha(Q,R))\leq$$

$$^{sup}_{\alpha\in\mathbb{A}^+}d^c_\alpha(P,Q)+^{sup}_{\alpha\in\mathbb{A}^+}d^c_\alpha(Q,R)=d^c(P,Q)+d^c(Q,R).$$

■